Modelling yields at the lower bound through regime shifts*

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Abstract

We propose a dynamic term structure model with stochastic regime switches to deal with the lower bound on nominal interest rates. A separate lower bound regime for the evolution of the state vector can capture the higher persistence of yields when monetary policy interest rates are constrained from below. State-dependent regime switching probabilities ensure that the likelihood of being in the lower bound regime increases, as interest rates fall closer to zero. We apply our model to U.S. data and show that it captures various properties of yields at the lower bound. Compared to the shadow rate approach, the regime-switching model suggests a much slower pace of normalization of U.S. policy interest rates. It also attributes non-zero probability to a "new normal" scenario characterized by a permanently lower long-run level of interest rates.

Keywords: zero lower bound; term premia; term structure of interest rates; monetary policy rate expectations; regime switches.

JEL Codes: E31, E40, E44, E52, E58, E62, E63.

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1 Introduction

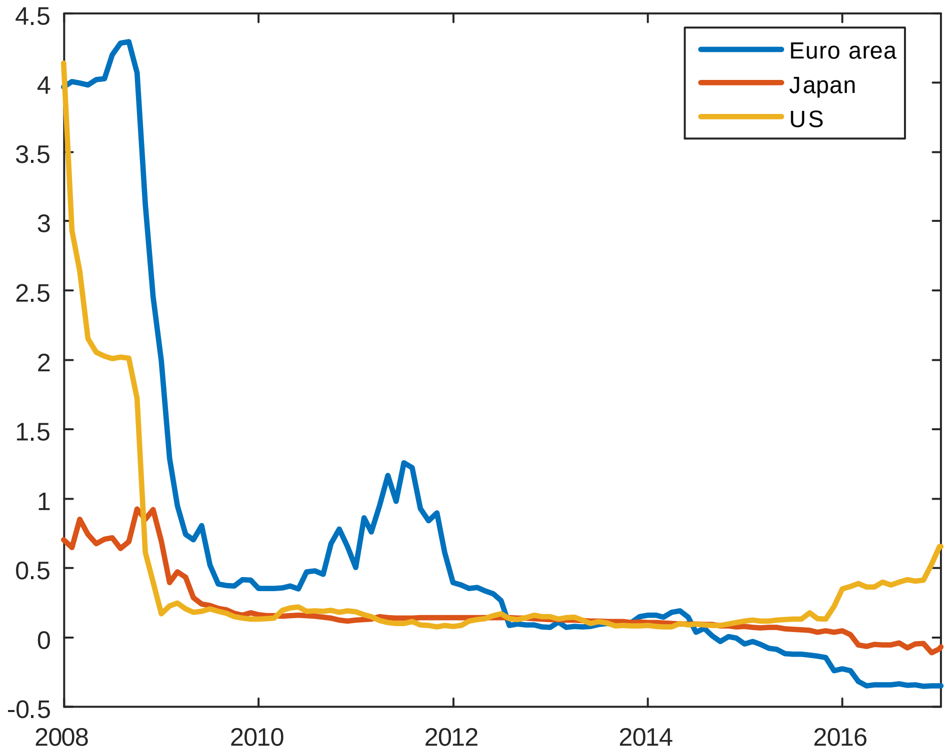
After the global financial crisis of 2008-09, short-term nominal interest rates have reached levels close to their lower bound (henceforth, LB) in many advanced countries – see Figure 1 for the experiences of the euro area, Japan and the United States. Over time two new lessons have been learned from this experience.

The first one is that, due to cash storage costs, the LB level is not literally zero. In theory a negative interest rate level on short-term bonds would give rise to an arbitrage opportunity – going short on the bond yielding a negative interest rate and long in cash. The zero level should therefore represent a lower bound for interest rates. In practice storage costs have discouraged banks from transforming central bank reserves into banknotes. In many constituencies, including the euro area, Sweden and Switzerland, key policy interest rates have became negative and have driven down with them short term market rates – see again Figure 1 for the euro area. As a results, a well-specified yield curve model need not exclude altogether, i.e. assign probability zero to, the possibility that yields may go slightly negative.

The second lessons of recent LB experiences is that periods of low interest (and inflation) rates are much more persistent after a recession in which they reach the lower bound, than after normal recessions. Excluding one main break over the 2006-2008 period, one-month money market rates have so far remained below the 25 basis points level for over 13 years in Japan. They have remained below that level for over 4 years in the euro area and for 7 years in the United States. This confirms theoretical arguments suggesting that periods when nominal rates reach their LB result in deeper and more prolonged recessions, and that they are characterized by different economic dynamics compared to "normal" situations, i.e. recessions where policy interest rates can be reduced without constraints (e.g. Eggertsson and Woodford, 2003). In other words, and again consistently with theoretical arguments, the evolution of the economy changes when interest rates are close to their lower bound. Monetary policy cannot impart sufficient economic stimulus, so the recovery is slower.

The aforementioned two lessons suggest that a regime-switching approach may be suitable to model yields at, and away from, the lower bound. This is the approach we adopt in this paper. Yield dynamics are explicitly described as a function of two possible policy regimes: a normal regime and a LB regime. Conditional on remaining in the normal regime, interest rates are an affine function of a Gaussian state vector, which follows an unrestricted VAR process.

Figure 1: One-month interest rates



Conditional on remaining in the LB regime, interest rates are again an affine function of a Gaussian state vector, but all parameters are allowed to differ from normal regime values and the dynamics of the state vector are restricted to imply that the short-term rate is a white noise process around a constant level. The most straightforward way to impose this restriction is to include the short rate in the state vector – a modelling device that we adopt throughout our analysis and which comes at virtually no loss of generality. Movements across the two regimes can occur stochastically and their probabilities are state-dependent. This implies that the probabilities to switch to the LB regime, conditional on being in the normal regime, can be high when short term rates are close to zero. The LB regime thus becomes increasingly likely to occur as short term interest rates approach the zero level.

This set-up has the advantage of allowing for both the aforementioned features of the recent, international experience with the LB. First, as soon as the system switches to the LB regime, the short-term policy rate will fluctuate around some LB level for as long as the economy stays in this regime. If this regime is persistent, other medium-term interest rates will also remain low, reflecting such expectations. Nevertheless, the LB regime does not literally prevent interest rates from falling below the LB on the policy rate, since risk premia can modify interest rate expectations under the risk-neutral probability measure. Moreover, short-term interest rates below the LB are also admissible due to the possibility of negative rate shocks. Second, the model can capture protracted periods of interest rates close to the LB, including "secular stagnation" scenarios of extremely persistent low growth, low inflation, and low nominal interest rates, because within-regime dynamics at the LB can be very different from those prevailing under normal circumstances. As long as the LB regime prevails, the model can therefore easily capture the phenomenon of the short rate remaining near zero for many years.

The main disadvantage of a regime switching model is that it leads to an increase in the size of the parameter vector to be estimated. This can be problematic when the sample size is limited. In our application, we mitigate this problem by relying on a model in which the state vector is made solely of observable variables. We also specify the functional dependence of regime-switching probabilities on the state vector on a priori grounds, rather than estimating its features from the data.

We estimate a version of the model on U.S. data relying on a yields-only model, where the state vector is made of the term spread, a measure of curvature, and the short rate. We show that this simple specification fits the data well. Across six nominal bond maturities, the estimated standard deviation of the measurement error is on average equal to 12 basis points. We also show that, in forecasting, the model effectively rules out the possibility of deeply negative rates. For example, in the middle of the period of policy interest rates at the effective lower bound, in late 2011, the forecast distribution from the model excludes interest rate values below -14 basis points.

Our estimates suggest that the U.S. economy was in the LB regime with a high probability from October 2008 until the end of 2015, and started moving towards the normal regime thereafter. At the end of our estimation sample, in April 2017, the probability of being in the normal regime returned to close to 100%. Nonetheless, forecasts beyond April 2017 indicate that the LB regime continues exerting an influence, pulling expected future yields downwards.

Regime-switching probabilities also have an effect on the decomposition of yields into expectations and risk premia components. At any point in time, risk premia do not only compensate investors for state risk, but also regime-switching risk. Regime-switching risk premia are near-zero before the Great recession, but they increase and hover around 10-30 basis points in monthly excess return terms when the economy is in the LB regime.

We also compare our results to those of a shadow rate model, which is a popular alternative to deal with the interest rate lower bound in term-structure modelling. The key difference between the two approaches emerges at the end of the sample. Since the short rate is no longer at the lower bound, the shadow rate model becomes akin to a standard Gaussian model and forecasts a normal return of yields to their long-run mean. In contrast, the regime-switching model forecasts a slower normalization of interest rates. Expected short-term interest rates are forecasted to reach around 3.7% in late 2018 in our version of the shadow rate model, while in the regime-switching model they remain well below 2% during this period.

Not only does the regime-switching model imply a slower normalization process; it can also capture changes in the future long-run level of nominal interest rates, e.g. a "new normal" characterized by a secular stagnation. This is a result of the assumption of state-dependent regime-switching probabilities. More specifically, we assume that the probability of returning to a normal regime depends on the level of short-term rates. It follows that the more persistently low such level is, the more likely a "new normal" future scenario consistent with the secular stagnation hypothesis. The higher short rates, the more likely a return to the pre-crisis normal.

Our paper is related to the applied literature studying the term structure of interest rate at the LB. This literature has developed fast over the past decade. The shadow rate model has proven to be the most successful empirical approach (see Bomfim, 2003, Ueno et al., 2006, Ichiue and Ueno, 2007, Kim and Singleton, 2012, and Christensen and Rudebusch, 2013, for estimates using Japanese yield data; Krippner, 2012, Ichiue and Ueno, 2013, Priebsch (2013) and Wu and Xia (2016) for application to the U.S.). Compared to standard single-regime affine models, it has two advantages (see also the discussion in Christensen and Rudebusch, 2013): it rules out negative interest rates (or rates below some other specified lower bound); and it can account for the observed reduction in the volatility of shorter-term yields when the policy rate is at the LB (see Swanson and Williams, 2014). The shadow rate model is also relatively parsimonious; away from the LB, it boils down to the standard affine formulation, which has been studied in an extensive literature.

A paper closer to our approach is Koeda (2013), which adopts a regime switching set-up in an application to Japan's experience with the lower bound. Nevertheless, some important differences characterize our approaches. First, Koeda (2013) assumes that the state dynamics under the risk-neutral measure are identical in the normal and the LB regimes, while we relax this assumption. Second, Koeda (2013) assumes that the regimes are observable not only to market participants, but also to the econometrician – an assumption we do not impose.

Our paper is organized as follows. Section 2 derives our term structure model with normal and lower-bound regimes. It derives approximate bond-pricing equations and then characterizes the model likelihood. An application of the model to U.S. data is described in Section 3. This section describes the fitting performance of the model and derives its implications for risk premia and for forecasting. The results of a comparison to the shadow rate model are also presented here. Section 4 concludes.

2 A regime-switching model of the lower bound

We start from the observation that a Gaussian dynamic term structure model has two shortcomings when short-term rates are closer to their lower bound. The first one, which is widely recognized, is to ignore the lower bound constraint, i.e. the fact that further reductions in short-term rates are low-probability events. The second shortcoming is that it assumes that the law of motion of the factors driving yields remains unchanged. This property is very unpalatable. If one takes a structural perspective on the dynamic of short-term, or "policy" interest rates, it becomes clear that the LB induces a nonlinearity in the law of motion of the state vector, not just a constraint on admissible yields. Intuitively, the short-term interest rate affects economic dynamics. When it can be reduced during a recession, the economy will recover at a normal speed. If however monetary policy is unable to lower it to the desired level, it cannot impart the desired monetary stimulus to the economy and the recovery will be slower.

Our regime-switching model captures the nonlinearity of economic dynamics at the lower bound in a flexible fashion. Compared to a Gaussian model, it can explicitly allow for changes in the law of motion of the factors driving the yield curve when the short term rate hits the LB. Compared to a shadow rate model, it relaxes the assumption that the state vector is expected to follow the same dynamic evolution, independently of whether the economy is, or is not, close to the LB. This assumption is likely to become more overly restrictive, the longer the length of the LB period – see also Svensson (2014).

Our starting point is a VAR model, under the objective probability measure $\mathbb{P}$, for the state vector x_t

$$x_{t+1} - \widehat{\mu}^j = \Phi^j \left(x_t - \widehat{\mu}^j \right) + \Sigma^j \varepsilon_{t+1} \tag{1}$$

where the state at time t is $s_t = j$, where either $j = N$ (for a Normal regime) or $j = L$ (for LB regime). This formulation can be easily generalized to accommodate additional discrete regimes. Moreover, the x_t vector can be expanded to include lags of its variables. Thus this formulation does not restrict us to work with a VAR(1) specification. Note that the conditional long-run means of the state vector $\widehat{\mu}^j$, the autoregressive coefficients Φ^j and the variance parameters Σ^j are all indexed by the prevailing regime. Equation (1) is equivalent to

$$x_{t+1} = \mu^j + \Phi^j x_t + \Sigma^j \varepsilon_{t+1} \tag{2}$$

for $\mu^j \equiv \left(I - \Phi^j \right) \widehat{\mu}^j$.

We can identify the LB regime by assuming that all entries in the row and column of Φ^L corresponding to r_t are equal to zero during the LB (this corresponds to the last row and column, since we order the variables in x_t such that the short-term rate r_t is the last element of the state vector). This implies that, in the LB regime, the short-term rate does not affect the other variables of the system and is itself an i.i.d. variable around a constant mean. We also assume that, at the LB, the short rate is not affected by shocks to the other equations. It follows that, conditional on remaining in the LB state, the law of motion of the short rate can be written as

$$r_{t+1} = \mu_r^L + \sigma_r^L \varepsilon_{t+1}^r$$

where μ_r^L is its conditional mean and σ_r^L is a scalar. In contrast, Φ^N, along with μ^N and Σ^N, are unrestricted in normal times.

The model is complemented by a short rate equation of the form

$$r_t = \delta_0 + \delta_x' x_t \tag{3}$$

where $\delta_0 = 0$ and δ_x is a vector of zeros, with the exception of a 1 (loading on the short rate) in the last position. Note that δ_0 and δ_x are regime-independent, but that differences in the mean short rate between regimes can be accommodated by μ^j.

Following Dai et al. (2007, henceforth DSY), assume finally for the SDF, $M_{t,t+1}$, that

$$\log M_{t+1} = -r_t - \Gamma_{t,t+1} - \frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\varepsilon_{t+1} \quad (4)$$

$$\Lambda_t^j = \lambda_0^j + \lambda_1^j x_t \quad (5)$$

$$\Gamma_{t,t+1} = \log \gamma_t^{j,k}, \quad (6)$$

where Λ_t^j are regime-dependent market prices of factor risk and $\Gamma_{t,t+1}$ are market prices of regime-switching risk. Non-zero Λ_t^j captures risk premia that compensate investors for unpredictable variation in the state variables, while non-zero $\Gamma_{t,t+1}$ captures premia required for being exposed to unpredictable regime shifts.

Note that DSY assumes that the market prices of factor risk are such as to produce a regime-independent feedback matrix under $\mathbb{Q}$, i.e. $\Phi^{N\mathbb{Q}} = \Phi^N - \Sigma^N\lambda_1^N = \Phi^L - \Sigma^L\lambda_1^L = \Phi^{L\mathbb{Q}}$. This assumption is unappealing for our application, because it would imply that bonds are priced *as if* the risk-neutral state vector dynamics in the lower bound regime were identical to those of the normal regime. We therefore allow for state dependent matrices $\Phi^{j\mathbb{Q}}$. A disadvantage with this approach is that closed-form solutions for arbitrage-free bond prices are unavailable. We therefore rely on the approximate bond pricing approach of Bansal and Zhou (2002). In the appendix we show, both analytically and by simulation, that the approximation is quite accurate for our parameter values.

Specifically, denote bond maturity by n and the price of an n-period bond at t by $P_{t,n}$, and note that the no arbitrage condition $P_{t,n} = \mathrm{E}_t\left[M_{t,t+1}P_{t+1,n-1}\right]$ can be rewritten as

$$1 = \mathrm{E}_t\left[M_{t,t+1}\frac{P_{t+1,n-1}}{P_{t,n}}\right].$$

In the appendix we prove the following proposition:

Proposition 1 *Bond prices can be written as*

$$P_{t,n} = \exp\left(-A_n^j - B_n^j x_t\right)$$

for

$$A_n^j = \sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\left(\delta_0^j + A_{n-1}^k + B_{n-1}^k\mu^{\mathbb{Q}j} - \frac{1}{2}B_{n-1}^k\Sigma^j\Sigma^j\left(B_{n-1}^k\right)'\right)$$

$$B_n^j = \sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\left(\delta_x' + B_{n-1}^k\Phi^{\mathbb{Q}j}\right)$$

starting from

$$\begin{aligned} A_1^j &= \delta_0^j \\ B_1 &= \delta_x' \end{aligned}$$

It follows that yields $y_{t,n} = \frac{1}{n}A_n^j + \frac{1}{n}B_n^j x_t$.

Before estimating the model, we need to specify the regime-switching probabilities under both $\mathbb{P}$ and $\mathbb{Q}$. Denote the transition probabilities from regime $s_t = j$ to regime $s_{t+1} = k$ as $\Pr[s_{t+1} = k | s_t = j] = \pi_t^{\mathbb{P}jk}$, for $0 \leq \pi_t^{\mathbb{P}jk} \leq 1$ and $\sum_{k=0}^{S} \pi^{\mathbb{P}jk} = 1$. In order to keep bond pricing somewhat tractable, we follow the existing asset pricing literature and assume that the $\mathbb{Q}$-probabilities, $\pi^{\mathbb{Q}jk}$, are constant over time. As for the $\mathbb{P}$-probabilities, $\pi_t^{\mathbb{P}jk}$, they can in general be time-varying and state dependent. More specifically, we model them using the cumulative probability of a multivariate normal distribution:

$$\pi_t^{\mathbb{P},NL} = \int^{\theta_x^{NL}} \frac{1}{\sqrt{(2\pi)^2 |\Sigma^N|}} \exp\left(-\frac{1}{2} \left(x - \mu_{t+1}^N\right)' \Sigma^{N^{-1}} \left(x - \mu_{t+1}^N\right) \right) dx,$$

where μ_{t+1}^N is the next-period conditional expectation of x, given that the economy is currently in state N, and where θ_x^{NL} is a vector of critical levels, or thresholds, that indicate at which point the sensitivity of the regime-switching probability is at its highest. In other words, these thresholds represent levels of the state variables where investors would start to become concerned that the economy would hit the LB.

The probability to leave the lower bound, $\pi_t^{\mathbb{P},LN}$, is modelled symmetrically, for thresholds θ_x^{LN}.

Finally, in order to link the $\mathbb{P}$ and $\mathbb{Q}$ transition probabilities, we follow DSY and assume that

$$\pi^{\mathbb{Q},NL} = \frac{\pi_t^{\mathbb{P},NL}}{\gamma_t^{NL}},$$

i.e. that the market prices of regime-switching risk are such that scaling the $\mathbb{P}$-probabilities by the risk price results in the $\mathbb{Q}$-probability. As shown by DSY, if regime shift risk is not priced, then the transition operabilities under $\mathbb{P}$ and $\mathbb{Q}$ would coincide.

2.1 Pricing consistency

Joslin, Singleton and Zhu (2011) and Hamilton and Wu (2012a) have highlighted an identification problem in affine term structure models, which arises from the property that the state vector x_t is often specified in terms of principal components of yields. In this section we discuss the problem in the context of our regime-switching model. The basic idea is to generalize to a regime-switching setting the approach of Ang, Piazzesi and Wei (2006).

The problem has to do with the prices of risk parameters λ_0^j and λ_1^j and it is particularly clear when the state vector is composed of yields only. As an illustrative example, which is also related to our application discussed in section 4, we focus on the simple case where the state vector is made of a slope factor s_t, given by the difference between a n-month maturity yield $y_{t,n}$ and the short rate r_t, and the short rate itself, so that $x_t^f = [s_t, r_t]'$.

Consider first the single-regime affine case. Given state x_t^f, short rate $r_t = [0,1]\ x_t^f$ and prices of risk $\Lambda_t = \lambda_0 + \lambda_1 x_t^f$, bond prices will follow a recursion such that $P_{t,k} = \exp\left(-A_k - B_k x_t^f\right)$. It is then immediately clear that estimation must be carried out under an additional constraint requiring that the slope factor $s_t = y_{t,n} - r_t$ is consistent with the pricing parameters in A_n and B_n. In this example, the constraints are $y_{t,n} = \frac{1}{n}\left(A_n + B_n\ x_t^f\right) = s_t + r_t$, or $\frac{1}{n}A_n = 0$ and $B_n = [n, n]$. More specifically note that in the single-regime affine case the bond recursions $A_n = A_{n-1} + B_{n-1}\mu^{\mathbb{Q}} - \frac{1}{2}B_{n-1}\Sigma\Sigma B_{n-1}'$ and $B_n = \delta_x' + B_{n-1}\Phi^{\mathbb{Q}}$ can be written explicitly as

$$
\begin{aligned}
A_n &= A_1 + \delta_x' \left(\sum_{j=1}^{n-1} \Psi_j\right) \mu^{\mathbb{Q}} - \frac{1}{2}\delta_x' \left(\sum_{j=1}^{n-1} \Psi_j \Sigma\Sigma \Psi_j'\right) \delta_x \\
B_n &= \delta_x' \left(\sum_{i=1}^{n} \left(\Phi^{\mathbb{Q}}\right)^{i-1}\right)
\end{aligned}
$$

for $\Psi_j \equiv \sum_{i=1}^{j} \left(\Phi^{\mathbb{Q}}\right)^{i-1}$. Assuming that the series $\sum_{i=1}^{n} \left(\Phi^{\mathbb{Q}}\right)^{i-1}$ converges, we can further write

$$
\begin{aligned}
A_n &= A_1 + \delta_x' \left(I - \Phi^{\mathbb{Q}}\right)^{-1} \left[\sum_{j=1}^{n-1} \left(I - \left(\Phi^{\mathbb{Q}}\right)^{j+1}\right)\right] \mu^{\mathbb{Q}} \\
&\quad - \frac{1}{2}\delta_x' \left(I - \Phi^{\mathbb{Q}}\right)^{-1} \left[\sum_{j=1}^{n-1} \left(I - \left(\Phi^{\mathbb{Q}}\right)^{j+1}\right) \Sigma\Sigma \left(I - \left(\Phi^{\mathbb{Q}}\right)^{j+1}\right)'\right] \left(I - \Phi^{\mathbb{Q}\prime}\right)^{-1} \delta_x \\
B_n &= \delta_x' \left(I - \Phi^{\mathbb{Q}}\right)^{-1} \left(I - \left(\Phi^{\mathbb{Q}}\right)^{n+1}\right)
\end{aligned}
$$

Under this assumption, the pricing consistency requirement in our example implies (note that by construction $A_1 = 0$)

$$\begin{aligned} \delta_x' \left(I - \Phi^{\mathbb{Q}}\right)^{-1} \sum_{j=1}^{n-1} \left[\left(I - \left(\Phi^{\mathbb{Q}}\right)^{j+1}\right) \mu^{\mathbb{Q}} - \frac{1}{2} \left(I - \left(\Phi^{\mathbb{Q}}\right)^{j+1}\right) \Sigma\Sigma \left(I - \left(\Phi^{\mathbb{Q}}\right)^{j+1}\right)' \left(I - \Phi^{\mathbb{Q}\prime}\right)^{-1} \delta_x \right] &= 0 \\ \delta_x' \left(I - \Phi^{\mathbb{Q}}\right)^{-1} \left(I - \left(\Phi^{\mathbb{Q}}\right)^{n+1}\right) &= [n, n] \end{aligned}$$

Recalling that, in this simple example, $\delta_x' = [0, 1]$, it is clear that the above constraints on A_n and B_n imply one nonlinear restriction on $\mu^{\mathbb{Q}} = \mu - \Sigma\lambda_0$ and two nonlinear restrictions on $\Phi^{\mathbb{Q}} = \Phi - \Sigma\lambda_1$. As a result, one element in λ_0 and one row of λ_1 cannot be freely estimated.

In the regime-switching case, the constraints must correspondingly ensure that bond prices are consistent with the state vector for both states N and L, i.e. for A_n^L, A_n^N, B_n^L and B_n^N. The recursions are more complex in this case, but in our example of this section they will imply restrictions on one row of λ_0^j and λ_1^j both under N and under L. As in Ang, Piazzesi and Wei (2006), we impose this type of restrictions in estimation.

2.2 Estimation

Estimation and inference are simplified by the lack of unobservable factors in the state vector x_t.

Define the yield on a zero coupon bond of maturity n at time t and in regime j as

$$y_{t,n}^j = \frac{1}{n} \left(A_n^j + B_n^j x_t\right)$$

and select m bond maturities for estimation.

Assume that all bonds except those used to construct the state variables are observed with measurement error. Conditional on $s_t = j$ and $s_{t+1} = k$, we can stack observed yields in a vector y_t so that

$$y_{t+1} = A^k + B^k x_{t+1} + \Sigma^m \varepsilon_{t+1}^m$$

where ε_{t+1}^m is a vector of measurement errors and, by assumption, Σ^m is diagonal and $corr\left(\varepsilon_{t+1}, \varepsilon_{t+1}^m\right) = 0$. This vector can be stacked with the (possibly expanded to account for additional lags) vector of state variables to write

$$\begin{bmatrix} y_{t+1} \\ x_{t+1} \end{bmatrix} = \begin{bmatrix} A^k + B^k \mu^j \\ \mu^j \end{bmatrix} + \begin{bmatrix} B^k \Phi^j \\ \Phi^j \end{bmatrix} x_t + \begin{bmatrix} \Sigma^m & B^k \Sigma^j \\ 0 & \Sigma^j \end{bmatrix} \begin{bmatrix} \varepsilon_{t+1}^m \\ \varepsilon_{t+1} \end{bmatrix}$$

or, for appropriately defined vectors W_{t+1}, $\overleftrightarrow{\mu}^{j,k}$, $\overleftrightarrow{\Phi}^{j}$, $\overleftrightarrow{\varepsilon}_{t+1}$ and matrix $\overleftrightarrow{\Sigma}^{j}$,

$$W_{t+1} = \overleftrightarrow{\mu}^{j,k} + \overleftrightarrow{\Phi}^{j} x_t + \overleftrightarrow{\Sigma}^{j} \overleftrightarrow{\varepsilon}_{t+1}$$

Conditional on x_t and on states $s_t = j$ and $s_{t+1} = k$, W_{t+1} is normally distributed with mean $\overleftrightarrow{\mu}^{j,k} + \overleftrightarrow{\Phi}^{j} x_t$ and variance-covariance matrix $\overleftrightarrow{\Sigma}^{j} \left(\overleftrightarrow{\Sigma}^{j}\right)'$, so that

$$f\left(W_{t+1}|x_t, s_t = j, s_{t+1} = k\right) = \mathbb{N}\left(\overleftrightarrow{\mu}^{j,k} + \overleftrightarrow{\Phi}^{j} x_t, \overleftrightarrow{\Sigma}^{j} \left(\overleftrightarrow{\Sigma}^{j}\right)'\right)$$

Note now that the joint probability $f\left(W_{t+1}, s_t = j, s_{t+1} = k|x_t\right)$ can be rewritten as

$$f\left(W_{t+1}, s_t = j, s_{t+1} = k|x_t\right) = f\left(W_{t+1}|x_t, s_t = j, s_{t+1} = k\right) f\left(s_t = j, s_{t+1} = k|x_t\right)$$

If we define the probability of regime $s_t = j$ given x_t as Q_t^j, i.e. $Q_t^j \equiv f\left(s_t = j|x_t\right)$, it follows that

$$\begin{aligned} f\left(s_t = j, s_{t+1} = k|x_t\right) &= f\left(s_t = j|x_t\right) f\left(s_{t+1} = k|x_t, s_t = j\right) \\ &\equiv Q_t^j \pi_t^{\mathbb{P}jk} \end{aligned}$$

and

$$f\left(W_{t+1}, s_t = j, s_{t+1} = k|x_t\right) = f\left(W_{t+1}|x_t, s_t = j, s_{t+1} = k\right) Q_t^j \pi_t^{\mathbb{P}jk}$$

so that the likelihood in period t can be obtained integrating out the discrete states

$$f\left(W_{t+1}|x_t\right) = \sum_{k=N,L} \sum_{j=N,L} f\left(W_{t+1}|x_t, s_t = j, s_{t+1} = k\right) Q_t^j \pi_t^{\mathbb{P}jk}$$

The sample likelihood is

$$\log L = \frac{1}{T-1} \sum_{t=0}^{T-1} \log f\left(W_{t+1}|x_t\right)$$

Denote the updated Q_t^j as $Q_{t+1}^k \equiv f\left(s_{t+1} = k|x_{t+1}\right)$. Then

$$Q_{t+1}^k = \frac{\sum_j f\left(W_{t+1}|x_t, s_t = j, s_{t+1} = k\right) Q_t^j \pi_t^{\mathbb{P}jk}}{f\left(W_{t+1}|x_t\right)}$$

3 An application to U.S. data

We test the performance of the model in an application to U.S. data.

The state vector is completely observable and includes three yield factors: a curvature factor c_t, defined as the sum of the 1-month nominal rate and the 10-year yield minus twice the 3-year yield, a slope factor s_t, defined as the spread between the 10-year and 1-month rate, and the 1-month nominal interest rate r_t. We estimate the model using six additional maturities: 3 and 6 months; 1, 2, 5 and 7 years. All the data is sampled at the monthly frequency (end-of-month values). The sample period runs from January 1987 to April 2017.

As discussed in the previous section, when estimating the model we place restrictions on the A_n^j and B_n^j coefficients associated to the 3-year and 10-year yields. Consistently with the definitions of the curvature and slope factors, the restrictions are on A_{36}^j, A_{120}^j, B_{36}^j and B_{120}^j and they ensure that $y_{t,36} = \frac{1}{36}\left(A_{36}^j + B_{36}^j x_t\right) = \frac{1}{2}(s_t - c_t) + r_t$ and that $y_{t,120} = \frac{1}{120}\left(A_{120}^j + B_{120}^j x_t\right) = s_t + r_t$. For $x_t \equiv [c_t, s_t, r_t]'$, this implies $A_{36}^j = 0$, $A_{120}^j = 0$, $B_{36}^j = 36\,[-1/2, 1/2, 1]$ and $B_{120}^j = 120\,[0, 1, 1]$.

Given the relatively short sample period and that presumably few regime switches occurred in the data, we adopt two simplifying assumptions.

First, we assume that the system was in the normal regime before 2008 and estimate a Gaussian term structure model over this period. This yields parameter estimates for μ^N, Φ^N and Σ^N, as well as $\mu^{\mathbb{Q}N}$, $\Phi^{\mathbb{Q}N}$, $\Sigma^{\mathbb{Q}N}$ plus A^N and B^N. We keep these parameters fixed when estimating the remaining ones.

Second, we assume that the transition probability from N to L, $\pi_t^{\mathbb{P},NL}$, is only a function of the short-term rate. This implies, intuitively, that the likelihood to switch to the lower bound regime is higher, the lower the level of the short-term rate. Specifically, we set

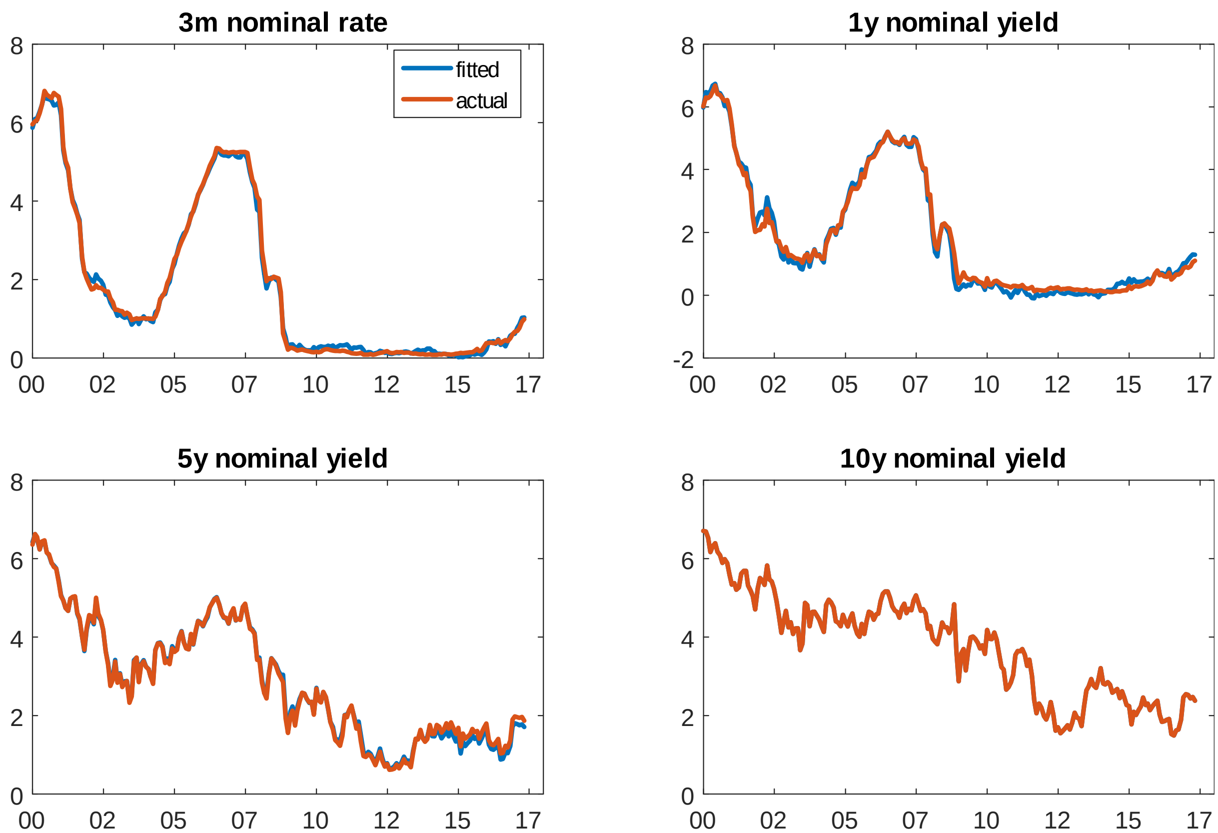
$$\pi_t^{\mathbb{P},NL} = \int^{\theta_r} \frac{1}{\sigma_r^N \sqrt{(2\pi)^2}} \exp\left(-\frac{1}{2}\left(\frac{r - \mu_{t+1}^{N,r}}{\sigma_r^N}\right)^2\right) dr.$$

We also assume symmetry in the probabilities to switch regime, so that $\pi_t^{\mathbb{P},LN} = 1 - \pi_t^{\mathbb{P},NL}$.

Given our short sample period, we dogmatically set θ_r to 12.5 basis points, corresponding to the 10th percentile of the unconditional distribution of the short rate over our estimation sample.

We estimate the other parameters as indicated in section 2.1. More specifically, we estimate μ^L, Φ^L, Σ^L, $\mu^{\mathbb{Q}L}$, $\Phi^{\mathbb{Q}L}$, $\Sigma^{\mathbb{Q}L}$ and the transition probabilities $\pi^{\mathbb{Q},NN}$ and $\pi^{\mathbb{Q},LL}$. As starting values, we use estimates of a Gaussian model estimated over the December 2008 - October 2015 period.

Figure 2: Actual and fitted yields

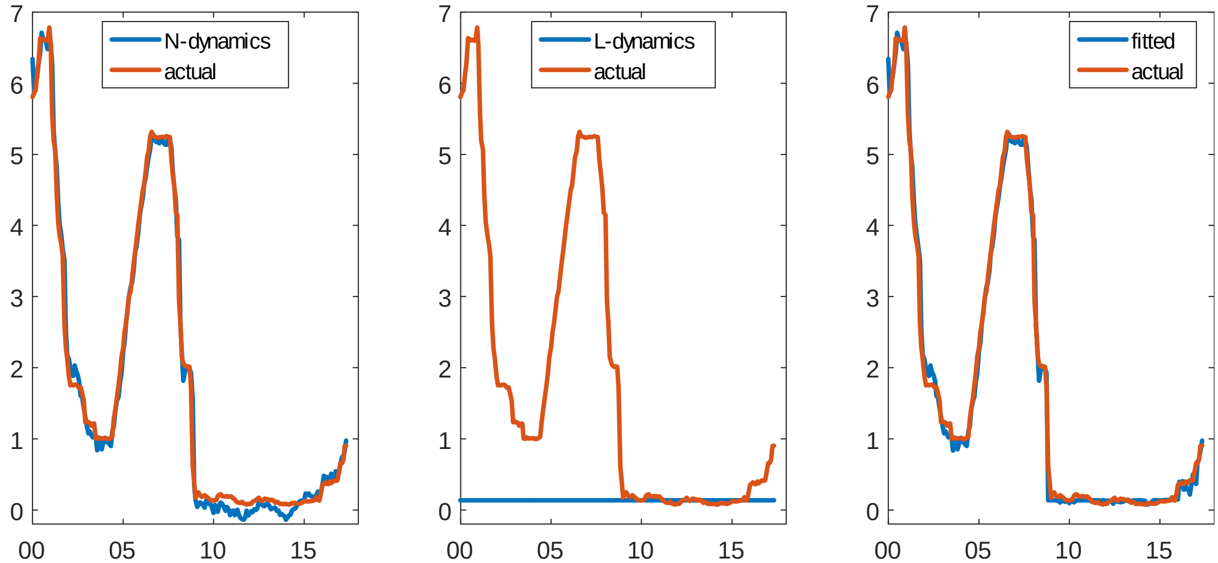


The parameter vector that maximizes the likelihood function is reported in Table 1. The common standard deviations of the yield measurement errors is around 12 basis points. Figure 2 reports actual and fitted values of bond yields across various maturities. Note that the 10y-yield is fitted perfectly by construction.

One noticeable feature of the parameter vector is the different steady state values conditional on each of the two regimes. In the lower bound regime, the steady state short rate is more than 3 percentage points lower (at 0.14 percent compared to 3.29 percent in the normal regime), the slope is around 80 basis points higher at 2.36% versus 1.58% percent, and the curvature factor is just below 0.5 (compared to almost 1 percent in the normal regime).

Figure 3 shows actual and fitted values of the short-term interest rate separately in the two regimes and in the full model. The figure highlights that the Gaussian model would not provide a terrible fit of the short term interest rate over the period of binding lower bound. For given N-regime parameters, the short-term rate would only dip into mildly negative territory,

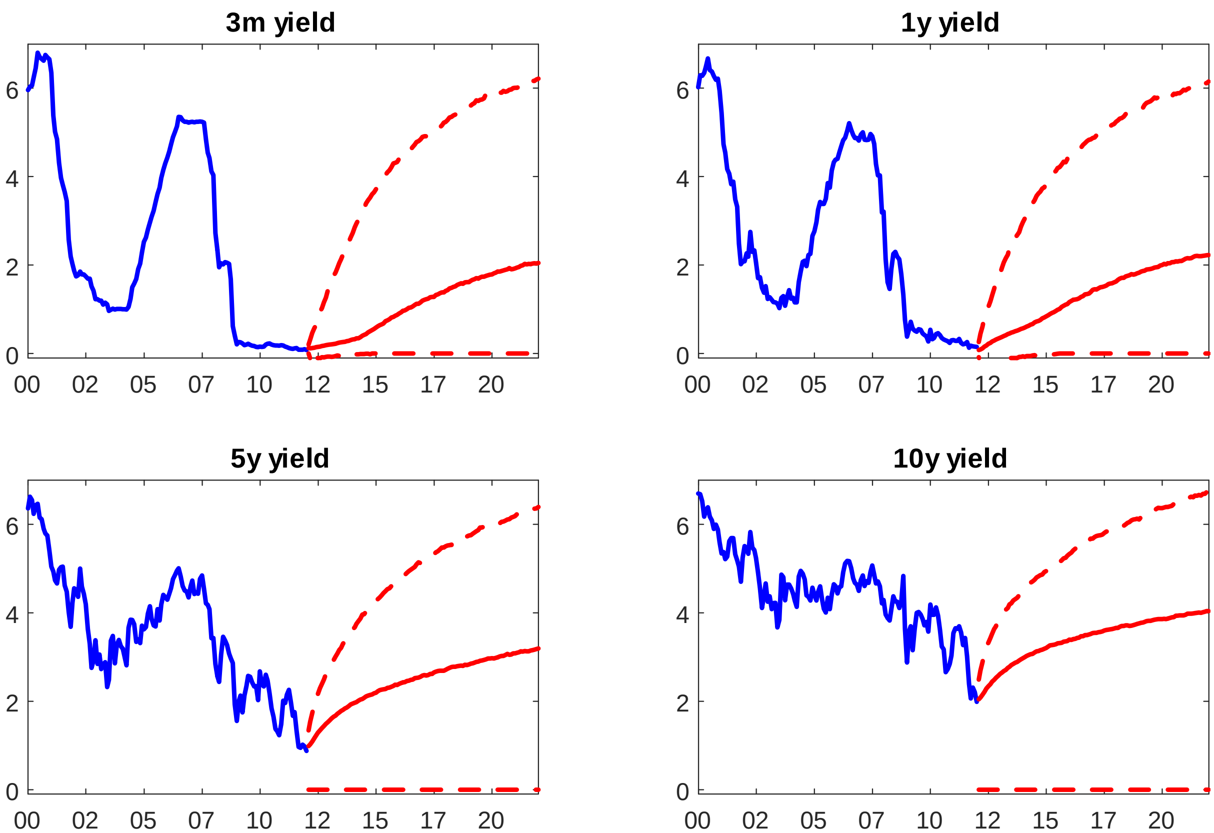
Figure 3: Actual and model short rate



hence fitting errors over 2009-2016 would be small, albeit highly persistent (see the left panel). By contrast, the ultra-low policy rates period can be captured by the L-regime parameters. As highlighted by the middle panel in the figure, however, the L-regime quickly becomes inconsistent with the data, once policy rates are above the level consistent with μ_L. The right panel in figure 3 shows that the regime switching model effectively combines the dynamics of the N and L regimes to ensure that the evolution of the short rate is consistent with the data over the whole sample period.

Figure 4 illustrates the model's ability to ensure that nominal interest rates do not attain deeply negative values. It shows the in-sample forecast distribution in late 2011, a time where the economy was in the lower bound regime with a probability close to 1. Going forward, the probability of moving to the normal regime was expected to slowly increases over time. Consequently, interest rates were slowly expected to increase too. Nevertheless, the forecast distribution is wide, and near-zero values remain possible for yields of all maturities. In all cases, however, the forecast distribution dips only very marginally into negative territory at times.

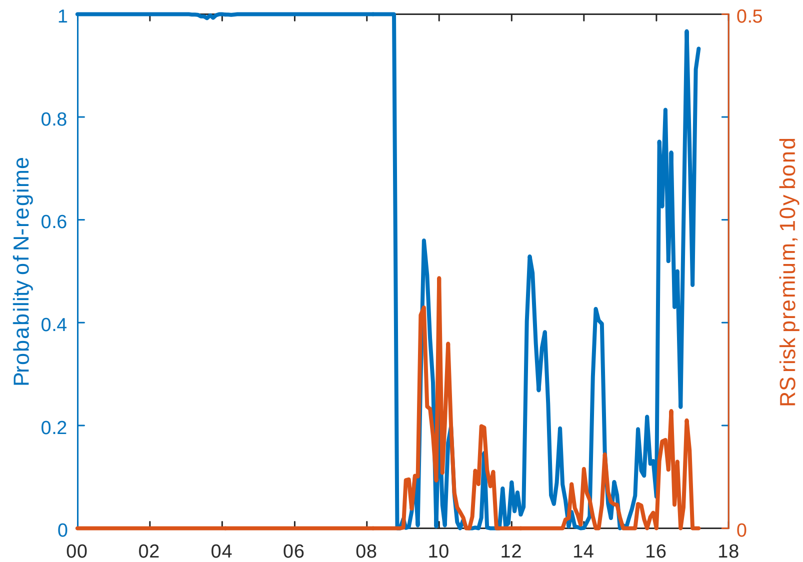
Figure 4: Yield forecasts as of end-2011



3.1 Inference on the regimes and implications for risk premia

Figure 5 reports the filtered probability of the economy being in the normal regime (blue line in the figure). The probability remains very close to one until the end of 2008, then it quickly drops to zero in October 2008, two months before the Federal Reserve established a target range for the federal funds rate of 0 to 1/4 percent. The probability of being in the normal regime registers some fluctuations until mid-2011, temporarily increasing towards the end of 2009, then falling again towards zero over 2010. As of August 2011, when the FOMC made its forward guidance more precise by stating that economic conditions would likely warrant that the federal funds rate remain exceptionally low "at least through mid-2013", the probability of being in the normal regime remains more narrowly close to zero. The probability then increases again at the time of the taper tantrums in mid-2013 and in spite of the fact that, in 2012, the FOMC had pushed the guidance on federal funds rates out to mid-2015. In early 2014, the

Figure 5: Probability to be in the normal regime and regime-shift risk premia



probability to be in the normal regime temporarily increases again, as the unemployment rate–one of the conditions for the state-contingent guidance introduced by the FOMC in December 2012 – fell faster than expected. Finally the probability increases again in December 2015, when the Committee decided to begin raising the target range for the federal funds rate from nearly zero. Nevertheless, the increase is initially mild. This is arguably consistent with the post-meeting statement, which indicated that the FOMC expected economic conditions to evolve in a manner consistent with *only gradual* increases in interest rates. After the new increase of the federal funds rate in December 2016, the probability of being in the normal regime increases more decisively, and by April 2017 it has returned to close to its pre-crisis levels.

Our model is also informative with regard to whether investors require compensation for being exposed to regime shift risk. We investigate this issue by examining one-period ahead expected excess returns conditional on assuming that factor risk is not priced. In other words, we set the market prices of factor risk (Λ_t^j) to zero and examine how model-implied expected excess returns behave over time. As shown in Appendix 2, any remaining non-zero expected excess returns in this scenario must (apart from a negligible Jensen's inequality term) be due to regime shift risk being priced, i.e. non-zero $\Gamma_{t,t+1}$. As shown in Figure 5 (red line), the results from this exercise indicate that regime shift risk premia on 10-year bonds remained very close to zero from the beginning our sample until early 2009, at which point they rise up to around 0.2% in monthly excess return terms. There appears to be a clear relationship between positive levels of regime-shift risk premia and the fall in the probability to be in the normal regime–see again Figure 5. Nevertheless, the relationship is not monotonic. For example, at the time of the taper tantrum in mid-2013 regime shift risk premia remain close to zero while the filtered probability to be in the normal regime increases to levels above 0.4.

All in all, these results suggest that regime-shift premia can be non-negligible around times when regime probabilities move significantly and investors become particularly concerned about the risk of a possible change in regime. Nevertheless, regime-shift premia are not a quantitatively crucial factor to explain yield dynamics over the time period we analyze.

Figure 6 shows term premia and implied expectations-hypothesis (EH) yields for bonds of various maturities. EH yields correspond to average expected short-term interest rates, i.e. yields net of the term premium component. The gist of these results is that fluctuations in EH yields explain most of the variation of actual yields for most maturities in the early 2000s, while fluctuations in risk premia are the main driving factor for observed yields during the period when the economy is in the lower bound regime.

3.2 Forecasting: a comparison to the shadow rate model

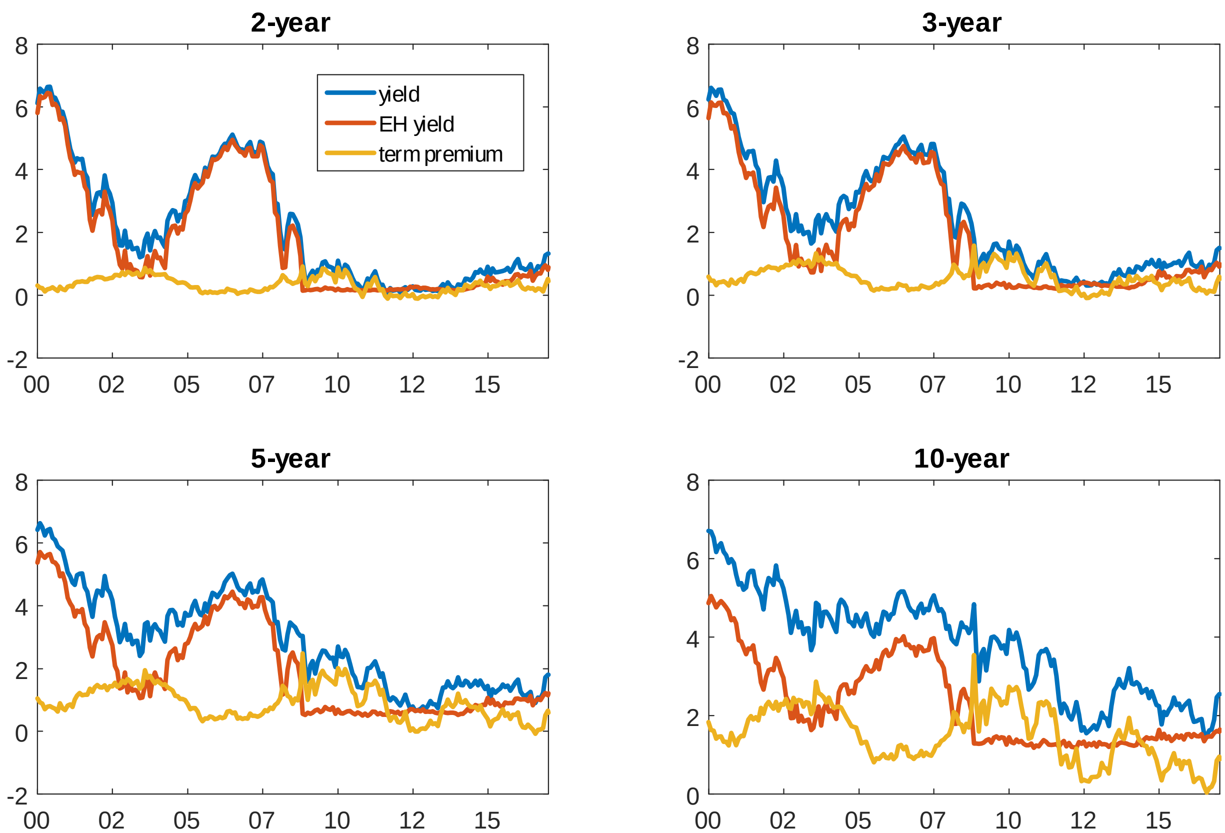
Figure 7 shows out-of-sample yield forecasts as of end-April 2017. The forecast horizon is 10 years.

Starting from the values near 1 in April 2017, the probability of being in the normal regime (not shown in the figure) is forecasted to fall slowly initially, reaching a trough of around 0.85% after 2 years, and then increase very slowly over the future 10 years. At the end of the forecast sample, the probability remains close to 0.9. As a result, the model also forecasts a very drawn out policy normalization process. Its mean prediction is that the short rate will reach values around 2.5% percent at the end of 2027, and that 10-year yields will just exceed the 4% mark.

At the same time, the forecast distribution is very wide. Figure 7 shows that the 90% uncertainty band for the 3-month rate in 2027 includes values as high as 6.5%, or as low a 0. The 10-year rate could reach almost 7% given this confidence level. These high levels of uncertainty are a reflection of persisting uncertainty over the probability of being in the normal regime.

By and large, the model forecast is consistent with the notion of *only gradual* increases in federal funds rates going forward. It also suggests that the new long-run value of nominal

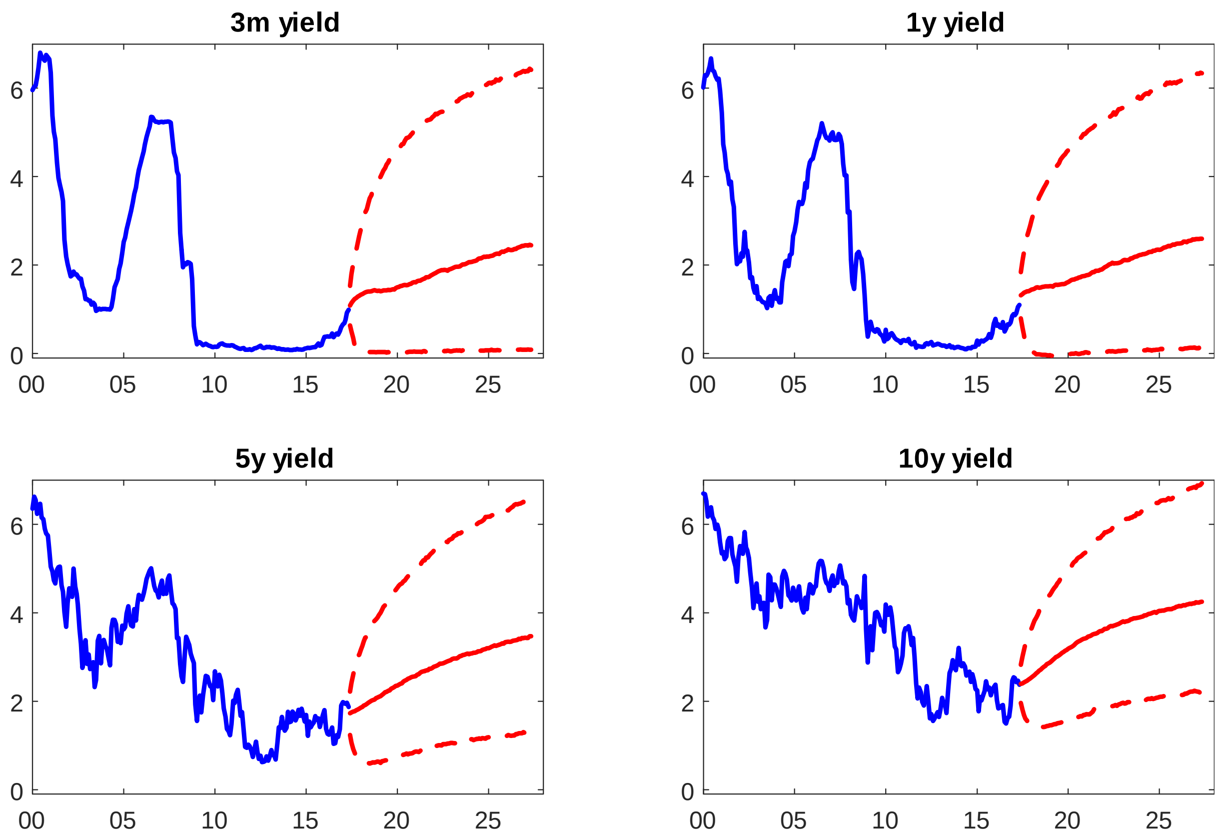
Figure 6: Term premia and expectations-hypothesis consistent yields



interest rates may in the future be lower than before the crisis. The recent period of persistently low real long-term rates can be interpreted as the beginning of a "new normal" characterized by a "secular stagnation" (see Summers, 2013), with permanently lower GDP growth and lower nominal and real interest rates. Some recent work has brought credence to the secular stagnation hypothesis. It has been shown that a downward adjustment in future long-run interest rates is likely to occur merely due to demographic factors (see Gagnon, Johannsen and Lopez-Salido, 2016). Simple observation shows that 94 months after the June 2009 trough, the current U.S. recovery already represents the third longest expansion in NBER records, but the effective federal funds rate remains below the 1% mark (by end-April 2017).

Our model can account for the new normal scenario through the time-varying conditional probability of changing regimes. The more persistently low short rates are, the more the model assigns a non-negligible probability to a version of the secular stagnation hypothesis. The higher short rates go, the more likely a return to the old normal.

Figure 7: Yield forecasts as of end-April 2017

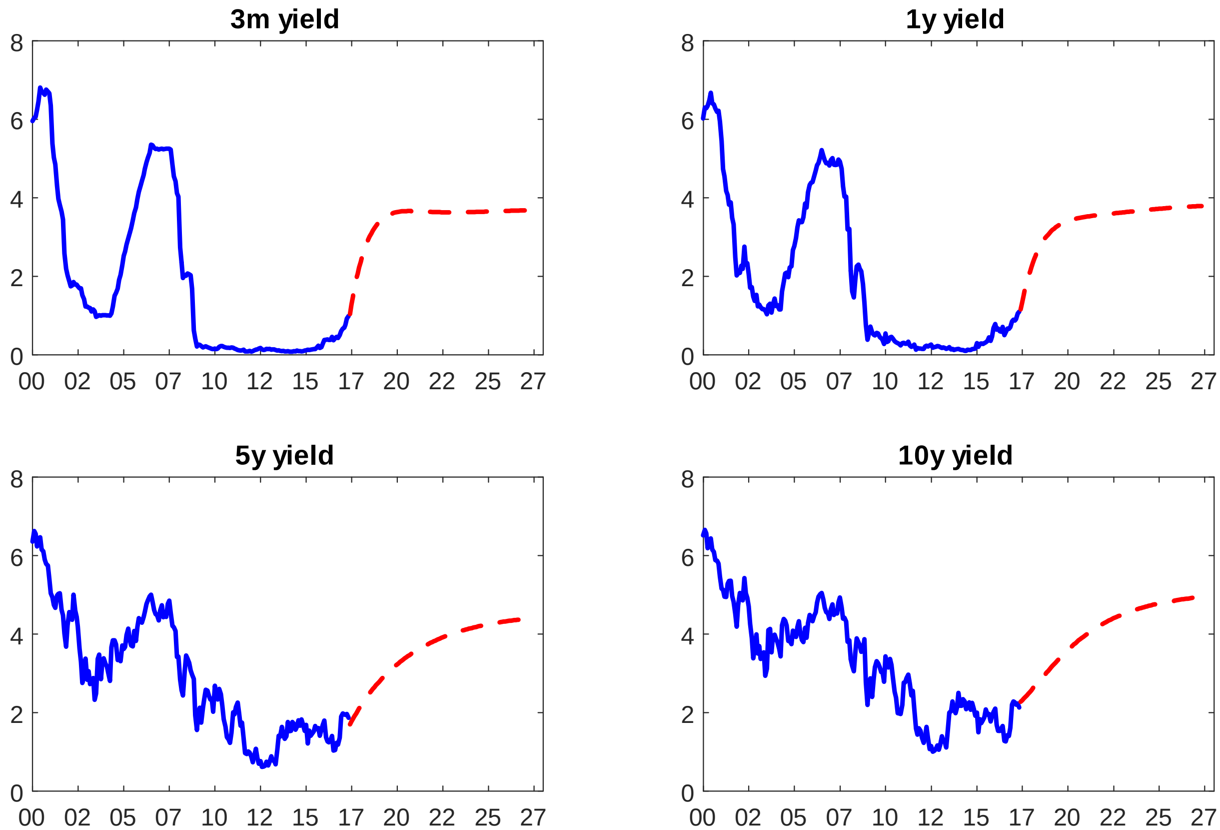


To highlight the ability of the regime-switching model to account for the special nature of recessions in which the economy reaches the lower bound regime, we compare its end-of-sample forecast to the forecast that would be obtained from a shadow rate model. For comparison purposes, we estimate a version of the shadow rate model over the same sample period. In so doing, we rely on the approach in Wu and Xia (2016), which is based on the use of forward rates (rather than yields) in estimation.[1]

The sample period is a key choice in the estimation of shadow rate models. Wu and Xia (2016) use a relatively short sample in which interest rates are often at the lower bound. As a result, the long-run mean of yields is estimated to be very low. For example, for short-term yields of maturity up to 1 year, the long-run mean is estimated to be below 1%. These values appear to be exceptionally low by historical standards, albeit not unreasonable in a

[1] More specifically, we use the Matlab code available on Cynthia Wu's webpage https://sites.google.com/site/jingcynthiawu/.

Figure 8: Yield forecasts from a version of the Wu and Xia (2016) shadow rate model



potential secular stagnation scenario. By contrast, Bauer and Rudebusch (2015) estimate an affine model over the pre-2008 period and then complement it with a shadow rate specification without re-estimating the parameter values. This approach is very efficient and illustrates a crucial advantage of the shadow rate model: it is very parsimonious, as it requires no additional parameters than an affine model. The Bauer and Rudebusch (2015) approach ensures that the long-run mean of interest rates is consistent with pre-crisis values. The paper also shows that this approach is not problematic for the model's ability to fit yields during the lower bound period. Outside the lower bound, the approach delivers the same properties as a standard affine model. More specifically, interest rates are assumed to return to normal levels as quickly as after a normal recession.

Forecasts from a version of the Wu and Xia (2016) model estimated over the pre-crisis sample are shown in figure 8. In contrast to figure 7, short term yields increase relatively quickly. In the Wu-Xia model, the 3-month rate is forecasted to reach levels close to the long-run level of around 3.7% by late 2018. The regime-switching model, meanwhile, has interest rates rising much more slowly, reaching only around 1.4% by end-2018.

All in all, the comparison of end-of-sample forecasts based on the regime-switching and shadow rate models suggest that the former model is potentially better able to capture the special nature of post-lower bound recoveries. The shadow rate model forces the researcher to specify at the outset whether the recovery will never be full – consistent with the secular stagnation hypothesis – or if it will be as fast as those following standard recessions. The regime-switching model weighs these two scenarios by the regime probabilities.

4 Conclusions

This paper analyzes a dynamic term structure model with stochastic regime switches to deal with the lower bound on nominal interest rates. We allow for separate laws of motion for the evolution of the state vector in normal times and at the lower bound. The model effectively minimizes the likelihood of interest rates becoming negative, even if it does not exclude this possibility altogether. It also captures the higher persistence of yields when monetary policy interest rates are constrained from below and monetary policy cannot impart sufficient stimulus to the economy.

In an application to U.S. data, the regime-switching model suggests a very slow pace of normalization of policy interest rates. It also attributes non-zero probability to a "new normal" scenario with a permanently lower long-run level of interest rates. Nevertheless, a high level of uncertainty characterizes these forecasts.

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Appendix

.1 Bond pricing using a log-linear approximation

Postulate that bond prices are exponentially affine in x_t

$$B_{t,n} = \exp\left(-A_n^j - B_n^j x_t\right)$$

Using the expression for bond prices and the stochastic discount factor, the no-arbitrage condition can be rewritten as

$$1 = \mathrm{E}_t\left[\exp\left(\begin{array}{c} -\delta_0^j - \delta_x' x_t - \frac{1}{2}\left(\Lambda_t^j\right)' \Lambda_t^j - \Gamma_t^{j,k} - A_{n-1}^k + A_n^j - B_{n-1}^k \mu^j - \left[\left(\Lambda_t^j\right)' + B_{n-1}^k \Sigma^j\right] \varepsilon_{t+1} \\ -\left(B_{n-1}^k \Phi^j - B_n^j\right) x_t \end{array}\right)\right]$$

Note that the independence between normal and regime-switching shocks implies that

$$\mathrm{E}_t\left[\exp\left(-\left[\left(\Lambda_t^j\right)' + B_{n-1}^k \Sigma^j\right] \varepsilon_{t+1}\right)\right] = \sum_{k=1}^{S} \pi_t^{\mathbb{P}jk} \exp\left(\frac{1}{2}\left(\Lambda_t^j\right)' \Lambda_t^j + B_{n-1}^k \Sigma^j \Lambda_t^j + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)'\right)$$

so that the no-arbitrage condition becomes

$$1 = \sum_{k=1}^{S} \pi_t^{\mathbb{P}jk} \exp\left(-\delta_0^j - \delta_x' x_t - \Gamma_t^{j,k} - A_{n-1}^k + A_n^j - B_{n-1}^k \mu^{\mathbb{Q}j} + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)' - \left(B_{n-1}^k \Phi^{\mathbb{Q}j} - B_n^j\right) x_t\right)$$

Finally use the assumption i.e. $\Gamma_t^{j,k} = \log \gamma_t^{j,k} = \log \frac{\pi_t^{\mathbb{P}jk}}{\pi^{\mathbb{Q}jk}}$ to obtain

$$1 = \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \exp\left(-\delta_0^j - \delta_x' x_t - A_{n-1}^k + A_n^j - B_{n-1}^k \mu^{\mathbb{Q}j} + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)' - \left(B_{n-1}^k \Phi^{\mathbb{Q}j} - B_n^j\right) x_t\right)$$

At this point, take a first oder approximation. Note that the right hand side must be 1 in steady state. The exponential can therefore be approximated around 0, using $\exp z \simeq 1 + z$. It follows that

$$0 = \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \left(-\delta_0^j - \delta_x' x_t - A_{n-1}^k + A_n^j - B_{n-1}^k \mu^{\mathbb{Q}j} + \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)' - \left(B_{n-1}^k \Phi^{\mathbb{Q}j} - B_n^j\right) x_t\right)$$

These equations are satisfied for all maturities as long as the A and B matrices follow the recursions

$$A_n^j = \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \left(\delta_0^j + A_{n-1}^k + B_{n-1}^k \mu^{\mathbb{Q}j} - \frac{1}{2} B_{n-1}^k \Sigma^j \Sigma^j \left(B_{n-1}^k\right)'\right)$$

$$B_n^j = \sum_{k=1}^{S} \pi^{\mathbb{Q}jk} \left(\delta_x' + B_{n-1}^k \Phi^{\mathbb{Q}j}\right)$$

starting from

$$\begin{aligned} A_1^j &= \delta_0^j \\ B_1 &= \delta_x' \end{aligned}$$

To compute the accuracy of the above approximation, we compare it to the exact solution for a few maturities.

Given the bond pricing equation

$$1 = \mathrm{E}_t \left[M_{t,t+1} \frac{B_{t+1,n-1}}{B_{t,n}} \right]$$

and the our assumptions on δ_0, δ_x', M_{t+1}, Λ_t^j and $\Gamma_{t,t+1}$, n-period bonds can be written exactly as

$$B_{t,n} = \sum_{k=S} \pi^{\mathbb{Q}s_t=j,s_{t+1}=k} \sum_{l=S} \pi^{\mathbb{Q}s_{t+1}=k,s_{t+2}=l} ... \sum_{z=S} \pi^{\mathbb{Q}s_{t+n}=y,s_{t+n+1}=z} \exp\left(-A_n^{j,k,l,m,...,z} - B_n^{j,k,l,m,...,z} x_t\right)$$

for

$$A_n^{j,k,l,m,...,z} = \delta_0 + A_{n-1}^{j,k,l,m,...,y} + B_{n-1}^{k,l,m,...,y} \mu^{\mathbb{Q}j} - \frac{1}{2} B_{n-1}^{k,l,m,...,y} \Sigma^j \Sigma^j \left(B_{n-1}^{k,l,m,...,y}\right)'$$

and

$$B_n^{k,l,m,...,z} = \delta_x' + B_{n-1}^{k,l,m,...,y} \Phi^{\mathbb{Q}j}$$

The above expression involves 2^{n-1} terms for a bond of maturity n, so it quickly becomes computationally intractable. We can however compute it quickly for bonds of up to 18-month maturity. The approximation error implied by the log-linear approximation for these maturities is always smaller than one tenth of a basis point.

.2 Regime shift risk premia

Conditional on the current regime $s_t = j$, the expected return on an n-period bond $P_{t,n}^j$ is

$$E_t^{\mathbb{P}} \left[\log\left(P_{t+1,n-1}\right) \mid s_t = j\right] - \log\left(P_{t,n}^j\right).$$

The current bond price is

$$P_{t,n}^j = \exp\left(-A_n^j - B_n^j X_t\right),$$

so the log price is

$$\begin{aligned}
\log\left(P_{t,n}^{j}\right) &= \log E_t^{\mathbb{Q}}\left[\exp\left(-r_t^j\right)P_{t+1,n-1} \mid s_t=j\right] \\
&= -r_t^j+\log\left(\sum_{k=1}^{S}\pi^{\mathbb{Q}jk}E_t^{\mathbb{Q}}\left[P_{t+1,n-1}^{k} \mid s_t=j\right]\right) \\
&= -r_t^j+\log\left(\sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\exp\left(-A_{n-1}^{k}\right)E_t^{\mathbb{Q}}\left[\exp\left(-B_{n-1}^{k}X_{t+1}\right) \mid s_t=j\right]\right) \\
&= -r_t^j+\log\left(\sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\exp\left(-A_{n-1}^{k}\right)\exp\left(-B_{n-1}^{k}\left(\mu^{\mathbb{Q}j}+\Phi^{\mathbb{Q}j}X_t\right)\right)E_t^{\mathbb{Q}}\left[\exp\left(-B_{n-1}^{k}\varepsilon_{t+1}\right) \mid s_t=j\right]\right) \\
&= -r_t^j+\log\left(\sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\exp\left(-A_{n-1}^{k}-B_{n-1}^{k}\left(\mu^{\mathbb{Q}j}+\Phi^{\mathbb{Q}j}X_t\right)\right)\exp\left(\frac{1}{2}B_{n-1}^{k}\Sigma^j\Sigma^jB_{n-1}^{k\prime}\right)\right) \\
&= -r_t^j+\log\left(\sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\exp\left(-A_{n-1}^{k}-B_{n-1}^{k}\left(\mu^{\mathbb{Q}j}+\Phi^{\mathbb{Q}j}X_t\right)+\frac{1}{2}B_{n-1}^{k}\Sigma^j\Sigma^jB_{n-1}^{k\prime}\right)\right)
\end{aligned}$$

The expected value ($\mathbb{P}$) of the $t+1$ log price is

$$\begin{aligned}
&E_t^{\mathbb{P}}\left[\log\left(P_{t+1,n-1}\right) \mid s_t=j\right] \\
=&\sum_{k=1}^{S}\pi^{\mathbb{P}jk}E_t^{\mathbb{P}}\left[\log\left(P_{t+1,n-1}^{k}\right) \mid s_t=j\right] \\
=&\sum_{k=1}^{S}\pi^{\mathbb{P}jk}\left(-A_{n-1}^{k}-B_{n-1}^{k}\left(\mu^{\mathbb{P}j}+\Phi^{\mathbb{P}j}X_t\right)\right).
\end{aligned}$$

The expected excess return is then

$$\begin{aligned}
xr_{t,n} &\equiv E_t^{\mathbb{P}}\left[\log\left(P_{t+1,n-1}\right) \mid s_t=j\right]-\log\left(P_{t,n}^{j}\right)-r_t^j \\
&= \sum_{k=1}^{S}\pi^{\mathbb{P}jk}\left(-A_{n-1}^{k}-B_{n-1}^{k}\left(\mu^{\mathbb{P}j}+\Phi^{\mathbb{P}j}X_t\right)\right)- \\
&\quad \log\left(\sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\exp\left(-A_{n-1}^{k}-B_{n-1}^{k}\left(\mu^{\mathbb{Q}j}+\Phi^{\mathbb{Q}j}X_t\right)+\frac{1}{2}B_{n-1}^{k}\Sigma^j\Sigma^jB_{n-1}^{k\prime}\right)\right).
\end{aligned}$$

Suppose factor risk is unpriced, $\Lambda_t=0$, so that $\mu^{\mathbb{Q}j}=\mu^{\mathbb{P}j}$ and $\Phi^{\mathbb{Q}j}=\Phi^{\mathbb{P}j}$, then we get

$$\begin{aligned}
xr_{t,n|\Lambda_t=0} &= E_t^{\mathbb{P}}\left[\log\left(P_{t+1,n-1|\Lambda_t=0}\right) \mid s_t=j\right]-\log\left(P_{t,n|\Lambda_t=0}^{j}\right)-r_t^j \\
&= \sum_{k=1}^{S}\pi^{\mathbb{P}jk}\left(-A_{n-1}^{k}-B_{n-1}^{k}\left(\mu^{\mathbb{P}j}+\Phi^{\mathbb{P}j}X_t\right)\right)- \\
&\quad \log\left(\sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\exp\left(-A_{n-1}^{k}-B_{n-1}^{k}\left(\mu^{\mathbb{P}j}+\Phi^{\mathbb{P}j}X_t\right)+\frac{1}{2}B_{n-1}^{k}\Sigma^j\Sigma^jB_{n-1}^{k\prime}\right)\right).
\end{aligned}$$

We can use this expression to measure the impact of priced regime shift risk in a world where factor risk is not priced. Only to the extent that regime shift risk is priced will $\pi^{\mathbb{Q}jk}$ differ from $\pi_t^{\mathbb{P}jk}$, resulting in a non-zero $xr_{t,n|\Lambda_t=0}$ (apart from a Jensen's inequality term). To see this, use the approximation

$$\log\left(\sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\exp\left(-A_{n-1}^{k}-B_{n-1}^{k}\left(\mu^{\mathbb{P}j}+\Phi^{\mathbb{P}j}X_t\right)\right)\right)\approx\sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\left(-A_{n-1}^{k}-B_{n-1}^{k}\left(\mu^{\mathbb{P}j}+\Phi^{\mathbb{P}j}X_t\right)\right),$$

and ignore the Jensen's inequality term to get

$$\begin{aligned} xr_{t,n} &\approx \sum_{k=1}^{S}\pi^{\mathbb{P}jk}\left(-A_{n-1}^{k}-B_{n-1}^{k}\left(\mu^{\mathbb{P}j}+\Phi^{\mathbb{P}j}X_t\right)\right)-\sum_{k=1}^{S}\pi^{\mathbb{Q}jk}\left(-A_{n-1}^{k}-B_{n-1}^{k}\left(\mu^{\mathbb{P}j}+\Phi^{\mathbb{P}j}X_t\right)\right) \\ &= \sum_{k=1}^{S}\left(\pi^{\mathbb{Q}jk}-\pi^{\mathbb{P}jk}\right)\left(A_{n-1}^{k}+B_{n-1}^{k}\left(\mu^{\mathbb{P}j}+\Phi^{\mathbb{P}j}X_t\right)\right). \end{aligned}$$

Table 1: Parameter values

For $x_t = \begin{bmatrix} c_t & s_t & r_t \end{bmatrix}'$ and $x_{t+1} - \widehat{\mu}^j = \Phi^j \left(x_t - \widehat{\mu}^j \right) + \Sigma^j \varepsilon_{t+1},\ j = N, L$

$$\Phi^{\mathbb{P}N} = \begin{bmatrix} \underset{(0.036)}{0.824} & \underset{(0.032)}{0.221} & \underset{(0.028)}{-0.197} \\ \underset{(0.024)}{-0.031} & \underset{(0.021)}{1.003} & \underset{(0.018)}{-0.013} \\ \underset{(0.015)}{-0.008} & \underset{(0.013)}{0.008} & \underset{(0.011)}{0.983} \end{bmatrix}$$

$$1200 \cdot \left(\widehat{\mu}^{\mathbb{P}N}\right)' = \begin{bmatrix} \underset{(0.333)}{0.473} & \underset{(1.153)}{1.581} & \underset{(1.673)}{3.286} \end{bmatrix}$$

$$1200 \cdot \Sigma^N = \begin{bmatrix} \underset{(0.276)}{0.342} & \underset{(0.096)}{-0.120} & \underset{(0.152)}{0.011} \\ & \underset{(0.161)}{0.282} & \underset{(0.072)}{-0.102} \\ & & \underset{(0.252)}{0.259} \end{bmatrix}$$

$$\Phi^{\mathbb{Q}N} = \begin{bmatrix} \underset{(0.387)}{0.751} & \underset{(0.320)}{-0.065} & \underset{(0.015)}{-0.037} \\ \underset{(-)}{0.262} & \underset{(-)}{1.027} & \underset{(-)}{0.029} \\ \underset{(-)}{-0.260} & \underset{(-)}{-0.017} & \underset{(-)}{0.970} \end{bmatrix}$$

$$1200 \cdot \left(\mu^{\mathbb{Q}N}\right)' = \begin{bmatrix} \underset{(0.389)}{0.320} & \underset{(-)}{-0.258} & \underset{(-)}{0.272} \end{bmatrix}$$

$$\Phi^{\mathbb{P}L} = \begin{bmatrix} \underset{(0.055)}{0.879} & \underset{(0.068)}{0.024} & 0 \\ \underset{(0.033)}{0.024} & \underset{(0.041)}{0.930} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$1200 \cdot \left(\widehat{\mu}^{\mathbb{P}L}\right)' = \begin{bmatrix} \underset{(1.481)}{0.962} & \underset{(1.896)}{2.361} & \underset{(0.073)}{0.136} \end{bmatrix}$$

$$1200 \cdot \Sigma^L = \begin{bmatrix} \underset{(0.226)}{0.207} & \underset{(0.066)}{0.009} & 0 \\ & \underset{(0.189)}{0.256} & 0 \\ & & \underset{(0.034)}{0.042} \end{bmatrix}$$

$$\Phi^{\mathbb{Q}L} = \begin{bmatrix} \underset{(0.696)}{1.233} & \underset{(0.301)}{0.037} & \underset{(0.113)}{-0.238} \\ \underset{(-)}{-0.216} & \underset{(-)}{0.926} & \underset{(-)}{0.228} \\ \underset{(-)}{0.219} & \underset{(-)}{0.083} & \underset{(-)}{0.771} \end{bmatrix}$$

$$1200 \cdot \left(\mu^{\mathbb{Q}L}\right)' = \begin{bmatrix} \underset{(0.288)}{-0.255} & \underset{(-)}{0.303} & \underset{(-)}{-0.292} \end{bmatrix}$$

$$1200 \cdot \sigma^m = \underset{(0.302)}{0.124}, \quad \pi^Q_{NN} = \underset{(0.033)}{0.989}, \quad \pi^Q_{LL} = \underset{(1.200)}{0.456}$$