Urban Structure, Land Prices and Volatility

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Abstract

We develop a dynamic general equilibrium model that describes the evolution of land prices and rental rates in a monocentric city. The model explores the implications of urban configurations that may differ in terms of the flexibility of the citys borders and land use, i.e., zoning and the presence of undevelopable land, as well as differences in transit technology, i.e., cars versus rail. The model also considers the effect of production technologies that have different land use intensities and agglomeration externalities. Our analysis suggests that volatility is amplified when production exhibits strong agglomeration effects, and is dampened when land use plays a larger role in the production function and when transit exhibits strong congestion effects. In some settings land supply constraints make rental rates more volatile. However, we also identify settings under which increases in land supply constraints dampen volatility.

1 Introduction

According to a research report by Savills, a UK real estate consultant, the total value of all real estate in the world is about US $217 trillion, which is about 2.7 times the world's GDP. Real estate is clearly the most important capital asset in the world economy, but as illustrated by the recent financial crisis, our understanding of the determinants of real estate valuation, and in particular, the volatility of real estate prices, is still incomplete. The existing discussion of real estate volatility in the academic literature is based on the idea that prices are expected to be more volatile in regions where the supply of real estate is relatively inelastic since, in these regions, shocks to demand are offset less by increases in supply.[1] The theoretical literature,

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[1]See, for example, Glaeser et al. (2006) and Hilber and Vermeulen (2016). For an excellent synthesis and review of this literature see Glaeser and Gottlieb (2009).

however, has not precisely articulated a mapping between urban characteristics, supply elasticities and price volatilities, making the empirical implementation of this idea somewhat less straightforward.[2]

This paper examines the relationship between urban characteristics and rent volatility within the context of the seminal monocentric city models originally developed by Alonso (1964), Mills (1967), and Muth (1969). In this class of models, all commercial activity is conducted in an exogenous central business district (CBD) at the center of the city and workers commute to work from the outer rings of the city. The productivity of the firms in the commercial sector, along with the cost of transporting workers from the outskirts to the CBD, determines rents, wages and the size of the city.

Our extension of these models explicitly considers agglomeration externalities that make firms more productive in larger cities, and thus amplify exogenous productivity shocks that attract additional workers to the city. We also consider alternative physical characteristics (or zoning restrictions) that effectively create land supply constraints, and as we show, dampen the effect of productivity shocks on population growth. We also explore the implications of transportation costs, which also effectively limit urban growth.

Consistent with the existing literature, supply constraints in our model always dampen the effect of exogenous productivity shocks on population growth. We show, however, that this does not necessarily imply that supply constraints make rents more sensitive to productivity shocks. Indeed, a key insight of our model is that when agglomeration externalities are sufficiently strong, supply constraints can dampen the effect of productivity shocks on rents. Specifically, since supply constraints dampen the population increase, they reduce the agglomeration channel that amplifies the productivity shock, and this in turn, dampens the increase in wages and rents. As we show, in addition to the magnitude of the assumed agglomeration parameter, the sensitivity of rents to exogenous productivity depends on the transportation cost function, the amount of land in the city that cannot be developed, land supply in the CBD, and the importance of land and capital in the firms' production function.

After analytically exploring how the response of wages and rents to productivity shocks depends on urban characteristics, we simulate a dynamic version of the model that allows us to quantitatively study land rent volatilities, serial correlation and rent to value ratios.[3]

[2]The approach taken in the empirical literature is to use measures of a city's terrain and land use regulation, e.g., Saiz (2010), to proxy for constraints that may influence supply elasticities.

[3]Although Berliant and Wang (2005) review a number of dynamic urban models, we believe that we are the first to study the fluctuation of rental rates. The focus of the existing literature is on capital accumulation and urban population growth rather than on how the design of cities affects the patterns of property prices and rents. There is clearly a relation between the growth rate of an urban economy and the growth rate of land rents and prices. However, as we show, the growth rate of the urban economy is just one of several determinants of the growth rate of rents.

Although the dynamic model assumes that the exogenous component of a city's productivity follows a random walk, we introduce persistence by assuming that agglomeration externalities are realized with a one period lag, i.e., total factor productivity in the current period is an increasing function of the city's population in the previous period. As we describe below, the lagged response to changes in total factor productivity is needed to generate the positively serially correlated rents as well as the dispersed rent to value ratios that we observe in the data.

The relation between land rent volatilities and urban characteristics generated by our dynamic model are consistent with the elasticities that we derived in our static model. In particular, our simulations reveal that rents in cities with stronger agglomeration externalities tend to be more volatile and that commercial rents in cities with greater residential supply constraints tend to be less volatile. The dynamic model also generates novel implications about serial correlations of land rents and rent to value ratios. As expected, since the persistence in our model is generated because of agglomeration externalities, we find that stronger agglomeration externalities tend to be associated with greater serial correlation. In addition, our simulation results indicate that land supply constraints always reduce the serial correlation of land rents. This is intuitive – the persistence is generated because population growth increases future productivity, so anything that dampens population growth dampens the magnitude of the serial correlation.

The dynamic model generates land rent volatilities and serial correlations that are roughly consistent with what we observe in data provided by CBRE, a commercial real estate brokerage and investment firm. Our simulations also allow us to explore differences in rent to value ratios across cities. To gauge the magnitude of these differences we assume that cities are initially identical and simulate exogenous productivity shocks and record the distribution of rent to value ratios generated after 50 years. We show that the distribution of rent to value ratios of commercial land is roughly consistent with the dispersion of rent to value ratios reported by Real Capital Analytics, a real estate data vendor.

As we mentioned at the outset, although our model builds on the traditional monocentric urban framework, we address issues that have attracted considerable attention in the recent literature. For example, there is a recent literature that explores the role played by housing supply constraints in the recent increase in the cross-city dispersion in housing prices. For example, Nieuwerburgh and Weill (2010) develops a dynamic general equilibrium model that illustrates how housing supply constraints can amplify relatively small differences in productivity and create relatively large differences in house prices. The Gyourko et al. (2013) model describes how supply constraints can further increase dispersion in housing prices if cities attract a heterogeneous mix of residents with different tastes in amenities, i.e., certain cities will have amenities that cater to a wealthier clientele who are willing to pay higher housing prices. Finally, Hseih and Moretti (2017) develop a model that illustrates how inelastic housing supplies, perhaps caused by restrictive zoning, dampened economic growth by implicitly limiting migration from less productive to more productive cities.

These more recent models extend the Rosen (1979) and Roback (1982) framework, which take the supply elasticities and productivity shocks as given, and ignore within city spatial characteristics and agglomeration externalities that can amplify and dampen exogenous productivity shocks. By including these elements in a monocentric city model we provide the micro-foundations of the cross-city differences in land supply constraints and productivity differences. As we show, these implications of these micro-foundations are not completely obvious. For example, Hseih and Moretti (2017) suggest a policy of developing public transportation to relax housing supply constraints in high productivity cities in order to reduce spatial misapplication. Our model explicitly addresses the role of transportation and shows that although better transportation increases migration to high productivity cities it does not necessarily dampen the effect of exogenous productivity shocks on housing prices.

The rest of the paper is organized as follows. Section 2 introduces the benchmark model and shows, that in general, the model exhibits multiple equilibria. Section 3 focuses on what we think is the most plausible equilibrium and examines the elasticities of wages, population and land rents with respect to changes in exogenous shocks to productivity. Section 4 considers these same elasticities in alternative settings that allow us to explore the implications of CBD land flexibility, boundary expandability and capital mobility. Section 5 introduces the dynamic model and shows the numerical results. Section 6 concludes and provides a discussion of potential future studies.

2 The Benchmark Model

In this section we develop our benchmark model. As we describe below, relative to existing monocentric urban models, the main contribution of the benchmark model relates city characteristics that are roughly related to the elasticity of the supply of land. Specifically, we consider the flexibility of city boundaries, the amount of land within the city that cannot be developed, and the transportation technology. Both capital and labor are assumed to be perfectly mobile in the benchmark model and the size of the CBD is exogenous. However, in extended models we consider endogenous CBD size and immobile capital.

2.1 Geometry of the City

The city consists of a commercial CBD of size S, implying a CBD radius of $\sqrt{S}/\sqrt{\pi}$, surrounded by rings of residential land indexed by i, with the ring nearest to the CBD being $i = 0$. The land area in each ring is normalized to unity and includes both usable and unusable land. Specifically,

Λ percent of the land cannot be developed, because of either geographical constraints, such as lakes or oceans, or regulatory constraints, such as green areas that are used for parks or drainage. In the benchmark model we will assume that these areas are evenly distributed throughout the city and have no inherent amenity values.

The distance from a ring to the CBD is measured by the distance between its inner circle to the circumferences of the CBD. Thus, for the i^{th} ring, the distance is the difference between the radius of its inner circle and the radius of the CBD. Since the inner circle of the i^{th} ring encompasses an area of $S+i$, its radius is $\sqrt{S+i}/\sqrt{\pi}$, hence its distance is

$$j_i = \frac{\sqrt{S+i} - \sqrt{S}}{\sqrt{\pi}} \tag{1}$$

The distance j_i is simply a non-linear transformation of the location index i, so without loss of generality, we use j to denote both distance and location, with $j = 0$ representing the inner-most ring with a zero distance. The outer-most ring, denoted by $j = J$, is endogenously determined by equating its rent with the exogenous agricultural rent.

2.2 Transportation Cost

If we denote w as the wage for all workers, the wage net of transportation costs for workers living at location j is $w \times e^{-f(j,N)}$ where N is the city population. The function $f(j,N)$ assumes that transportation costs increase with wages, since transit takes time, and also increases with population, since larger cities are more congested.[4] Specifically, the transportation cost function $f(j,N)$ is assumed to have the following form:

$$f(j,N) = \beta_0 + \beta_1 j + \beta_2 jN \tag{2}$$

where $\beta_1 > 0$ is the distance gradient of transportation, and $\beta_2 > 0$ captures the congestion effect. The congestion effect increases with distance since

$$\frac{\partial f(j,N)}{\partial N} = \beta_2 j. \tag{3}$$

The net wage (wage net of transit cost) for workers living in location j is

$$W(j) = w \times e^{-f(j,N)}$$

2.3 Firms and Workers

The city is populated by a continuum of firms and a continuum of workers. Both are price takers and produce tradable goods, which serve as the numeraire in the model. Following the

[4]The exact transportation cost as a fraction of wage is $1 - e^{-f(j,N)}$. We call $f(j,N)$ the transportation cost function since $1 - e^{-f(j,N)} \approx f(j,N)$ when $f(j,N)$ is small.

standard practice in the urban literature, we assume the land and capital are owned by absentee owners who collect rent from either land or capital but do not live in the city.

2.3.1 Workers

Workers are perfectly mobile both within and across cities, which implies that they realize a reservation level of utility. Each worker is endowed with one unit of labor and allocates their wage to land rent, transit cost, and the consumption good. Workers have the option to live adjacent to the CBD and have zero commuting costs, or alternatively, they can live farther-out and spend resources to commute to the CBD.

Workers at location j take their wage and the land rent as given and choose their consumption of land, h, and the consumption good, c to solve the following optimization problem:

$$\max_{c,h} = u(c,h)$$

$$s.t.$$

$$c + p_r(j)h = w \times e^{-f(j,N)} \tag{4}$$

where $p_r(j)$ is the rental rate of residential land in location j.

It is straightforward to show that the optimal allocation between land and the consumption good satisfies:

$$p_r(j) = \frac{\partial u(c,h)/\partial h}{\partial u(c,h)/\partial c} \tag{5}$$

The right side of the above equation is the marginal rate of substitution between land and the consumption good. Given the assumed Cobb-Douglas utility function, i.e., $u(c,h) = c^{1-\theta}h^{\theta}$, equation (5) becomes:

$$p_r(j) = \frac{\theta}{1-\theta}\frac{c}{h}. \tag{6}$$

From equation (6), we get $c = \frac{1-\theta}{\theta}p(d)h$. Substituting this into the budget constraint (equation (4)) yields the optimal consumption good choice,

$$c(j) = (1-\theta)w \times e^{-f(j,N)} \tag{7}$$

and land demand function

$$h(j) = \theta\frac{we^{-f(j,N)}}{p_r(j)} \tag{8}$$

Since workers are identical, the rents, in equilibrium, make workers indifferent about where they live. Because rents decrease with distance to the CBD, workers that live near the CBD consume less land but more of the consumption good.

2.3.2 Firms

There exists a unit measure of identical firms that use land in the city's CBD along with capital and labor to produce the consumption good using a constant returns to scale Cobb-Douglas production function:

$$F(\ell, k, n) = A\ell^{\sigma} k^{\xi} n^{1-\sigma-\xi} \tag{9}$$

where ℓ, k and n are land, capital and labor input respectively, the relative importance of which is determined by the share parameters σ, ξ, and $1-\sigma-\xi$, respectively. A is the total factor productivity (TFP) of this city relative to other cities.

The firms take productivity A, land rent p_c, the price of capital r, and wage w as given, and solve the following optimization problem:

$$\max_{\ell,k,n} F(\ell, k, n) - wn - rk - p_c\ell$$

subject to equation (9). From the first-order conditions, we obtain the usual allocation rules as the following:

$$\frac{\ell}{n} = \frac{\sigma}{1-\sigma-\xi}\frac{w}{p_c} \tag{10}$$

$$\frac{k}{n} = \frac{\xi}{1-\sigma-\xi}\frac{w}{r} \tag{11}$$

$$\frac{\ell}{k} = \frac{\sigma}{\xi}\frac{r}{p_c} \tag{12}$$

2.4 The Equilibrium

We start this subsection by presenting the partial equilibrium bid-rent function, which describes the rental rate as a function of the wage and the distance from the CBD. We then describe the general equilibrium, which determines wages, the rent in the CBD, the amount of capital deployed and the population of workers in the city.

2.4.1 Bid-rent Functions

Following Fujita (1989) and Lucas and Rossi-Hansberg (2002), we separately describe bid-rent functions for the residential and commercial land markets. These functions describe the market clearing land rents for given levels of reservation utility, the wage rate, and the rental price of capital.

Residential Bid-rent Functions By substituting equation (7)-(8) into the Cobb-Douglas utility function, we can express the worker's reservation utility as a function of rent, the wage rate and transportation costs:

$$\underline{u} = \frac{(1-\theta)^{1-\theta}\theta^{\theta}}{p_r(j)^{\theta}} we^{-f(j,N)}$$

which can be rearranged as follows:

$$\begin{aligned} p_r(j) &= \left[\frac{(1-\theta)^{1-\theta}\theta^{\theta}}{\underline{u}} we^{-f(j,N)}\right]^{1/\theta} \\ &= B_0 \left[we^{-f(j,N)}\right]^{1/\theta}, \end{aligned} \tag{13}$$

where $B_0 = \left(\frac{(1-\theta)^{1-\theta}\theta^{\theta}}{\underline{u}}\right)^{1/\theta}$ decreases with the reservation utility.

This residential bid-rent function expresses rent as a function of exogenous reservation utility $\underline{u}$. The bid-rent function also reveals a positive relation between the residential land rent and the endogenous wage. Notice wage net of transportation cost $we^{-f(j,N)}$ is raised to the power of $1/\theta > 1$, which implies that a 1% increase in net wages causes a more than 1% increase in rent. This follows because workers, with higher wages and even higher land rent, substitute some land consumption for non-land consumption to achieve the reservation level of utility.

Commercial Bid-rent Function Because firms enter and exit the city freely, owners of commercial land take all the economic benefits from production. Thus commercial land rent equals the maximum revenue from one unit of land after paying for labor and capital. Production per unit of land is $f(\ell) = Ak^{\xi}n^{1-\sigma-\xi}$, thereby the commercial bid-rent function is solved from:

$$p_c = \max_{n,k} Ak^{\xi}n^{1-\sigma-\xi} - wn - rk.$$

First-order conditions with respect to labor and capital are:

$$\begin{aligned} wn &= (1-\sigma-\xi)Ak^{\xi}n^{1-\sigma-\xi} \\ rk &= \xi Ak^{\xi}n^{1-\sigma-\xi} \end{aligned}$$

We substitute these first-order conditions back into the problem of maximizing revenue minus wages and capital rents per unit of land. After some algebra we obtain

$$p_c = \left[\frac{A\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}{r^{\xi}w^{1-\sigma-\xi}}\right]^{\frac{1}{\sigma}} \tag{14}$$

Notice that p_c is a decreasing function of the wage w, which is in contrast to residential rents, p_r, which increases with w. Ceteris paribus, higher wages allow workers to pay more for rent, but reduce the rent that firms can pay and still earn zero profits. Similarly, commercial rent is lower when the capital price is higher. It is also easy to see that commercial land rent increases with productivity A.

2.4.2 City Level Variables

In addition to rents and wages, a city, in this model, is characterized by its total factor productivity, its population, its physical size, and the total amount of capital it rents.

Aggregate Quantities The relationships between wages, aggregate capital, and population are derived from the first order condition of firms. Since firms are identical, equations (10)-(12) hold for each firm. Thus at the city level we have:

$$\frac{N}{S} = \frac{1-\sigma-\xi}{\sigma} \times \frac{p_c}{w}, \tag{15}$$

$$\frac{N}{K} = \frac{1-\sigma-\xi}{\xi} \times \frac{r}{w}, \tag{16}$$

The above quantities are derived from the firms' optimization problem and thus reflect the number of workers per unit of land and per unit of capital demanded by firms. Capital is elastically supplied at a fixed cost, so its price and aggregate quantity is determined completely by the firm's demands. Labor is also elastically supplied, but at an exogenous reservation utility level rather than an exogenous wage rate. Because rents and transportation costs increase as the city grows, the wage rate must increase as the supply of labor grows. For a given rent gradient and wage rate we can determine the amount of land consumed per worker in location j, $h(j)$, from equation (8). Since there $1-\Lambda$ units of developable land at each location, the number of residents per location is $(1-\Lambda)/h(j)$. Integrating over all locations, we derive the total number of workers in the city as a function of the wage rate and the rent gradient:

$$N = \int_{j=0}^{J} \frac{1-\Lambda}{h(j)} dj = \int_{j=0}^{J} \frac{(1-\Lambda)p_r(j)}{\theta w e^{-f(j,N)}} dj \tag{17}$$

Since the right hand side of this equation, the number of workers at each location, is determined in part by congestion, as expressed by the transportation cost function $f(j, N)$, N is on both sides of the equation. Hence, N must be solved as the fixed point that satisfies both sides of the equation. As we will show, for any given wage rate, the above equation implies a unique population, i.e. a unique fixed point.

The equilibrium quantity of residential land is uniquely determined by J, the distance from the city's border and the CBD. Equating the residential bid-rent function at location J and the exogenous agricultural $\underline{p}$, the equilibrium boundary satisfies:

$$\underline{p} = p_{r(j=J)} = B_0 \left[w e^{-f(J,N)} \right]^{1/\theta}, \tag{18}$$

which is equivalent to $f(J, N) = log(w) - \theta log\left(\frac{\underline{p}}{B_0}\right)$. Using the explicit form of transportation cost function, we obtain

$$J = \frac{log(w) + \theta log\left(\frac{B_0}{\underline{p}}\right) - \beta_0}{\beta_1 + \beta_2 N} \tag{19}$$

City Level TFP While individual firms take the city TFP as given, the equilibrium TFP is determined endogenously as a function of the population. We assume the production externality takes the following form:

$$A = \tilde{A}N^{\lambda} \tag{20}$$

where $\tilde{A}$ is the exogenous productivity of a city, and λ is the agglomeration parameter that determines to what extent the city TFP increases with total number of employment in the city.

Table 1 lists parameters of the model along with some key variables.

Table 1: Parameters and Some Key Variables

λ	agglomeration effect
ξ	capital share in production
σ	land share in production
θ	land share in preference
$\tilde{A}$	exogenous productivity
$f(j,N)$	transportation cost, $f(j,N) = \beta_0 + \beta_1 j + \beta_2 jN$
β_1	distance gradient of $f(j,N)$
β_2	congestion effect or population gradient of $f(j,N)$
Λ	fraction of unusable land in each residential location
S	CBD size
J	city boundary
$\underline{u}$	reservation utility
B_0	$\left(\frac{(1-\theta)^{1-\theta}\theta^{\theta}}{\underline{u}}\right)^{1/\theta}$

2.4.3 General Equilibrium

With the rental price of capital exogenously given, we need to determine three equilibrium prices: $\{p_r, p_c, w\}$. As given in equation (13), residential land rent in each location is pinned down by p_r, the rent at location $j = 0$, and transportation costs. These variables, along with $\{N, K, J, A\}$, represent the seven endogenous variables that are determined by the seven equations, i.e. equations (13)-(20). In Appendix A.1 we show that given any $\{w, N\}$ pair, each of the remaining endogenous variables is uniquely determined and the system of seven equations can be reduced to the following two equations that describe wages and population:

$$log(N) = \frac{1}{\lambda-\sigma} log\left(\frac{r^{\xi}}{\tilde{A}\xi^{\xi}(1-\sigma-\xi)^{1-\xi}S^{\sigma}}\right) + \frac{1-\xi}{\lambda-\sigma} log(w) \tag{21}$$

$$log(N) = log\left(\frac{(1-\Lambda)B_0}{\theta}\right) + \frac{1-\theta}{\theta} log(w) + log\left(\int_0^J e^{-\frac{1-\theta}{\theta}f(j,N)} dj\right) \tag{22}$$

Equation (21) is the aggregate labor demand equation, which describes total labor demand in a city when the commercial land market is in equilibrium and the agglomeration effect is taken into account.[5] Obviously the equation represents a linear relationship between $log(w)$ and $log(N)$.

Equation (22) is the aggregate labor supply equation. For a given wage, it describes the number of workers that are housed in the city when the residential land market clears.[6]. As we show in Appendix A.2, the equation represents a non-linear relationship between $log(w)$ and $log(N)$. Its slope converges to infinity as population tends toward zero, and the slope converges to $\frac{1-\theta}{2\theta}$ as population tends toward infinity.

It is noteworthy that the slope of equation (22) is smaller when the transportation cost is larger. To see this, we define the following function :

$$F = \frac{\theta}{1-\theta}\left(1 - e^{-\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J}\right)\left(\frac{\beta_1 + 2\beta_2 N}{\beta_1 + \beta_2 N}\right) \tag{23}$$

Here F is a transformation of transportation costs between the CBD and the periphery. It is increasing in both N and J since both parenthesized terms in equation (23) are increasing in N and J. F converges to zero as N and J tend toward zeros. As N tends toward infinity, the limit of F is $\lim_{N\to\infty} F = \frac{2\theta}{1-\theta}$. When N and J are not too large, it is well approximated by $F \approx (\beta_1 + 2\beta_2 N)J$

It is easy to verify the following relationship between the slope of equation (22) and the transportation cost:

$$\frac{dlog(N)}{dlog(w)} = \frac{1}{F}. \tag{24}$$

Therefore, the slope is large when the transportation cost is small, and a small increase in wage will enable the city to accommodate a lot more population. When the transportation cost gets large, the slope becomes smaller and the city needs to increases the wage to a larger extent in order to accommodate additional population.

Obviously F becomes a constant and equation (22) represents a linear relationship provided that (i) $\beta_2 = 0$, i.e., higher population does not cause transportation congestion and (ii) J is exogenously specified. Under these conditions, our model is a special case of Lucas and Rossi-Hansberg (2002), and it has a unique equilibrium as both the aggregate demand function and aggregate supply function are linear functions.

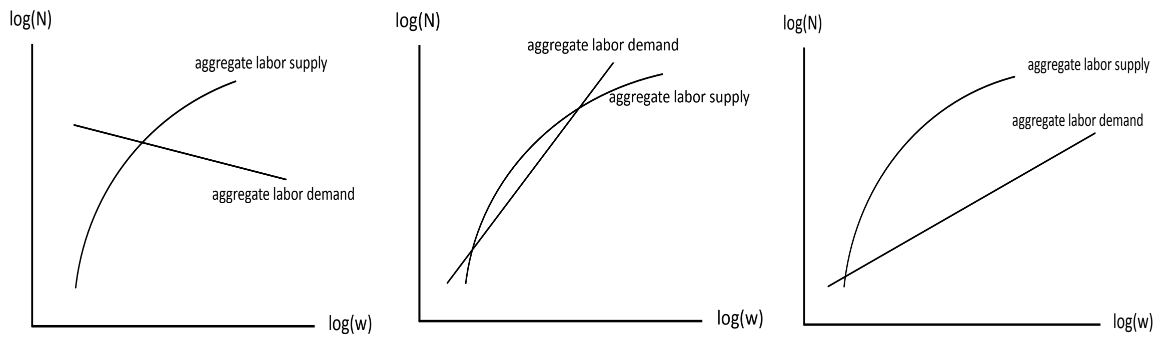
However, the equilibrium is not necessarily unique in our more general setting that allows the physical size of the city to be endogenous and assumes that rising population causes more

[5] Our "aggregate labor supply function" and "aggregate labor demand function" correspond to the "population supply function" and "population demand function" in Fujita (1989).

[6] The upper bound of integral in equation (22) is the city boundary J which is also a function of wage and population as shown in equation (19)

congestion. Figure 1 illustrates three possibilities.[7] The first panel illustrates a case where the agglomeration effect is relatively weak and there is a unique equilibrium. As illustrated in the second panel, when the agglomeration effect is stronger, there can be two equilibria. In the small city equilibrium firms are less productive and thus pay lower wages, but workers are able to achieve their reservation utility levels because rents and congestion are lower in smaller cities. In this case with multiple equilibria, workers in small and large cities achieve the same reservation utility and firms all make zero profits. However, landlords make more money in large cities. Finally, the third panel illustrates the case where the agglomeration externality is very high. In this case, we still get a small city equilibrium. However, a sufficiently large city will generate a level of utility for workers that is greater and the level of utility will grow without bounds as the size of the city increases.

Figure 1: Equilibrium



(a) $\sigma > \lambda$ (b) $\sigma < \lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta}$ (c) $\lambda \geq \sigma + (1-\xi)\frac{2\theta}{1-\theta}$

Note: The number of equilibrium (equilibria) is determined by the slopes of aggregate labor supply curve and aggregate labor demand curve.

These three possibilities are summarized in the following Proposition. The formal proof is given in Appendix A.2.

Proposition 1 *Given the exogenous transportation function $f(j, N)$ and the exogenous levels of reservation utility $\underline{u}$, productivity $\tilde{A}$, rental rate of capital r, and CBD size S, the model*

1. *has a unique equilibrium if $\lambda < \sigma$;*
2. *has two equilibria if $\sigma < \lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta}$.*
3. *has two possibilities: (i) an equilibrium with a small population, and (ii) a situation where there is no steady-state, and the size of the city explodes, if $\lambda > \sigma + (1-\xi)\frac{2\theta}{1-\theta}$.*

[7]We deviate from the convention of using the vertical axis for prices for the sake of simpler mathematical expression in the equations.

In our analysis below, we impose regularity conditions that rule out some perverse outcomes. First of all, we assume $\lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta}$, thereby rule out the explosive city as illustrated in panel (c) of the above figure.

Second, for the case of $\lambda > \sigma$, the condition $\frac{dlog(N)}{dlog(w)} < \frac{1-\xi}{\lambda-\sigma}$ is imposed. This rules out the small city equilibrium when the equilibrium is not unique, because it guarantees the aggregate labor supply curve is flatter than aggregate labor demand curve at the point they intersect.

Note that the small city equilibrium is not stable. Starting from this equilibrium, a positive productivity shock causes firms to hire more workers and pay higher wage, and the migration of more workers lead to even higher productivity. This feedback loop continues until the economy reaches the large city equilibrium in panel (b) of figure 1, and it continues indefinitely in panel (c).

The two **regularity conditions** to be used in the rest of the paper is summarized below:

$$\lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta} \tag{25}$$

$$\frac{1}{F} < \frac{1-\xi}{\lambda-\sigma} \qquad , when \quad \lambda > \sigma. \tag{26}$$

Focusing on the large city stable equilibrium, we get the following corollary, which describes how the population of the city is affected by the exogenous specification of the size of the CBD, the productivity $\tilde{A}$, and the proportion of the land in the city that can be developed:

Corollary 1.1 *In a non-explosive equilibrium,*

1. *both the wage rate and population increase with the CBD size S;*
2. *population decreases with the share of undevelopable land; wage decreases (increases) with the share of undevelopable land if $\lambda > \sigma$ ($\lambda < \sigma$).*

Item 1 the corollary is easily seen from equation (21) and figure 1: the larger S is reflected in the rightward shift of the aggregate labor demand curve, leadings to a larger population and a higher wage. For item 2, notice that more undevelopable land is reflected in the rightward shift of the aggregate labor supply curve, which leads to a smaller population and a lower (higher) wage if $\lambda > \sigma$ ($\lambda < \sigma$). Intuitively, cities will have larger populations when there is more land available for both the commercial and residential sectors.

3 Comparative Statics

This subsection examines how land rent, wages and population are affected by exogenous productivity changes. Specifically, we will analyze what we refer to as productivity elasticities, defined as the rate of change of an endogenous variable in response to an exogenous productivity shock. We will pay particular attention to how these elasticities are affected by alternative city structures, like the transportation technology and exogenous restrictions on development.

We use $\zeta_w = \frac{dw/w}{d\tilde{A}/\tilde{A}}$, $\zeta_N = \frac{dN/N}{d\tilde{A}/\tilde{A}}$, and $\zeta_{p_c} = \frac{dp_c/p_c}{d\tilde{A}/\tilde{A}}$ and $\zeta_{p_r(j)} = \frac{dp_r(j)/p_r(j)}{d\tilde{A}/\tilde{A}}$ to denote the productivity elasticity of wage, population, the commercial land rent and the residential land rent in location j respectively. These elasticities, which are essentially comparative statics between two steady states, provide intuition about the volatility of these variables. In section 5, we simulate a multi-period version of this model with i.i.d. shocks to the exogenous part of TFP. There the volatilities of the endogenous variables are directly analogous to the elasticities we derive describe in this section.

3.1 The Elasticity of the Wage, Population and City Boundary

We will start by examining the elasticity of the wage rate and the population. As discussed by Glaeser et al. (2006) and others, a positive shock to productivity is likely to result in a large increase in population and a small increase in wages in a city that can easily expand and a small increase in population and a large increase in wages in a city whose growth is constrained. In our model, the size of the city is not explicitly constrained, but workers bear higher transportation costs when the population increases, and these costs effectively constrain the size of the city.

To explore the tradeoff between population growth and wage growth in our model, we differentiate the aggregate labor supply function (equation 22) with respect to the productivity shock, $log(\tilde{A})$, to obtain the following relationship between the elasticity of the wage rate and the elasticity of the population.[8]

$$\frac{\zeta_w}{\zeta_N} = F \tag{27}$$

The above equation, where F is defined in equation (23), indicates that a productivity shock affects wages relatively more than population when the cost of traveling between the CBD and the boundary is higher.

By differentiating equation (21), the aggregate labor demand equation, with respect to $\tilde{A}$, we obtain

$$\zeta_N = -\frac{1}{\lambda - \sigma} + \frac{1-\xi}{\lambda - \sigma}\zeta_w. \tag{28}$$

The above equation, together with equation (27), leads to the following expression for the elasticity of population:

$$\zeta_N = \frac{1}{-\lambda + \sigma + (1-\xi)F} \tag{29}$$

[8]Equation (27) can also be easily derived from equation (24).

Since we have ruled out the small city equilibrium using inequality (26), the denominator in the above equation is positive, which implies that $\zeta_N > 0$. Given the relation between ζ_w and ζ_N in equation (27), $\zeta_N > 0$ implies that $\zeta_w > 0$. Therefore, both population and the wage rate increase (decrease) in response to a positive (negative) change in the exogenous productivity parameter $\tilde{A}$.

We also study how the city boundary responds to a productivity shock. It seems intuitive that shocks that increase population are likely to increase the physical size of a city, but as we show, this is not always the case. From the equation $J = \frac{log(w)+\theta log\left(B_0/p\right)-\beta_0}{\beta_1+\beta_2 N}$ (equation 19), we see that a positive productivity shock affects the city boundary through two channels: (i) congestion costs increase as population increases, which increases the transportation costs of living far from the CBD; and (ii) a higher wage allows workers to spend more on transportation from the periphery, while keeping their utility at the reservation utility level. As shown in Appendix A.3, the derivative of $log(J)$ with respect to $log(\tilde{A})$ which is elasticity of city boundary with respect to $\tilde{A}$ is

$$\zeta_J = \frac{F - \beta_2 JN}{(\beta_1 + \beta_2 N)J}\zeta_N \tag{30}$$

Therefore $\zeta_J > 0$, i.e., increased productivity expands the size of the city, if and only if

$$F - \beta_2 JN > 0, \tag{31}$$

for which a sufficient condition is

$$(\beta_1 + 2\beta_2 N)J < \frac{2\theta}{1-\theta} \tag{32}$$

See Appendix A.3 for the proof.[9]

It follows that when transportation costs, as captured by the left-side of inequality (32), are low relative to workers' preference for land (θ), a positive productivity shock causes the boundary of the city to expand. However, when transportation costs are relatively high and workers' preference for land is relatively low, condition (31) does not hold, and a positive productivity shock causes the city boundary to contract despite the rising population. Intuitively, if the growth in population leads to very high congestion costs, and if workers do not have a strong preference for land consumption (i.e. θ is small), then workers will respond to the worse congestion by moving closer to the CBD, decreasing the area of the city and increasing its density.

We summarize the above results in the following proposition.

Proposition 2 *Given a positive (negative) productivity shock,*

[9] Recall that F is bounded above by $2\theta/(1-\theta)$ thus it might be smaller than $\beta_2 JN$ which is unbounded.

- *the population and the wage rate always rise (declines),*
- *the city boundary J expands (contracts) if and only if condition* (31) *holds.*

In the simulations that we describe in Section 5, we find that condition (31) is satisfied under what we consider reasonable parameter values.[10] Based on this, we simplify the following analysis by assuming that these conditions are satisfied, implying that we focus only on cities that expand geographically in response to a positive productivity shock.

3.2 Elasticity of Residential Land

We now turn to the elasticity of land rent, the main focus of this paper. From the residential bid-rent function (equation 13), we derive the following equation,

$$\zeta_{p_r(j)} = \frac{1}{\theta}\zeta_w - \frac{\beta_2 jN}{\theta}\zeta_N. \tag{33}$$

which shows that residential land rent elasticity increases with ζ_w but decreases with ζ_N. In other words, a positive productivity shock leads to an increase in the wage rate, which has the effect of increasing rent. However, the increase in productivity also increases population, and thus congestion, which at least partially offsets the advantage of the higher wage.

Using equations (27) and (29) to substitute out ζ_w and ζ_N in (33), we derive the following expression for the elasticity of residential land rent:

$$\zeta_{p_r} = \frac{1}{\theta} \times \frac{F - \beta_2 jN}{-\lambda + \sigma + (1-\xi)F} \tag{34}$$

where the denominator is positive since we rule out the small city equilibrium, and the numerator is positive since we assume condition (31) always holds. Under these conditions, $\zeta_{p_r} > 0$, which means that a positive (negative) productivity shock causes land rent to rise (fall) in every location in the city.[11]

A few points are clear from this equation. First, ζ_{p_r} increases with λ, which means that land rent is more sensitive to productivity shocks in cities with stronger agglomeration externalities. This again reflects the fact that agglomeration externalities amplify productivity shocks. Second, ζ_{p_r} increases with ξ, the share of capital in the production function, but ζ_{p_r} decreases with σ, the share of land. What this means is that if land is used less as a production input, its value

[10] In the simulation exercise in Section 5, our calibration leads to $(\beta_1 + 2\beta_2 N)J \approx 0.15$ and $\frac{2\theta}{1-\theta} \approx 0.86$.

[11] While the above result is intuitive, it should be stressed that there are conditions under which rents in some parts of the city decline as the population increases. Indeed, in cases where the area of the city declines as the population increases, rents near the boundary of the city will decline in response to positive productivity shocks.

is more sensitive to productivity shocks. This is intuitive: since the supply of commercial land is fixed, and capital is elastically supplied, the value associated with an increase in productivity is captured by the land holders. Hence, if very little land is needed in the production process, the value of land, per unit, will increase more when productivity increases. As we will show in Section 4, in the case where capital is also in fixed supply, ζ_{p_r} decreases with both ξ and σ, i.e., land rent elasticity is dampened when capital is less important in the production process.

Equation (34) also shows that land rent elasticity is location-specific. Close-in locations are more sensitive to productivity shocks than farther-out locations. This is due to our assumption that farther-out locations are more affected by congestion. Indeed, land rent elasticity is the same in each location if the congestion has the same effect on commuting times in each location. Specifically, given the alternative transportation cost function of $f(j, N) = \beta_0 + \beta_1 j + \beta_2 N$, we have

$$\zeta_{p_r^\star} = \frac{1}{\theta} \times \frac{F^\star - \beta_2 N}{-\lambda + \sigma + (1-\xi)F^\star} \tag{35}$$

which is not location specific. Appendix A.4 provides the proof as well as the expression of $F^\star$ as a function of total population and city boundary.

Determining the relation between transportation technology and the land rent elasticity in equation (33) is less straightforward since the transportation cost F shows up in both the denominator and the numerator. By taking the partial derivative of ζ_{p_r} with respect to F, we can show that land rent elasticity decreases with F if and only if $\lambda - \sigma > (1-\xi)\beta_2 jN$.

Land rent elasticity also depends on city population. There are two reasons for this. As equation (33) shows, since larger cities are more sensitive to increased congestion, land rent is more sensitive to increased population, as captured by the term $\beta_2 jN$. In this channel the effect of population on land rent elasticity is location-specific. There is a second channel: a larger population implies a larger transportation cost F. As just discussed, the effect of the second channel depends on how large the agglomeration effect is. For cities with strong agglomeration, both channels predict that a larger population leads to a smaller land rent elasticity. For cities where the agglomeration effect is below certain threshold, we can show that a larger population leads to the larger land rent elasticity. The formal proof is presented in Appendix B.1.

We summarize our analysis of how residential land rent elasticities are influenced by city characteristics in the following proposition:

Proposition 3 *In the benchmark model, residential land rent elasticity is*

1. *always positive.*
2. *increasing in λ and ξ but decreasing in σ in each location.*
3. *decreasing in distance to the CBD.*

4. *decreasing in F if $\lambda - \sigma > (1-\xi)\beta_2 jN$; and increasing otherwise.*

5. *decreasing in N if $\lambda - \sigma > -\chi$; and increasing otherwise.*

where $\chi = \frac{F - \frac{dF}{dlog(N)}}{\frac{dF}{dlog(N)} - \beta_2 jN}(1-\xi)\beta_2 jN \approx \frac{(1-\xi)\beta_1 j}{2-j/J}$.[12] Note that residential land rent elasticity is more likely to decrease with N than to decrease with F, since the former requires $\lambda - \sigma > -\chi$ which is a much less strong condition than $\lambda - \sigma > (1-\xi)\beta_2 jN$, the condition for the latter.

It should be noted that the term $\lambda - \sigma$ plays an important role in item 4-5 in the above proposition. The agglomeration parameter, λ, and σ, the parameter that represents the importance of land in the production function, have offsetting effects when population grows. Specifically, λ determines the extent to which productivity per worker increases when population increases, while σ determines the extent to which productivity per worker falls due to the reduced amount of commercial land per worker. In our model, $\lambda - \sigma$ determines the net effect of these two offsetting forces on productivity per worker, and hence captures the magnitude of the net agglomeration externality.

We also study how S, the size of the CBD, affects land rent elasticity. It is noteworthy that S does not show up in equation (34), so it affects rent elasticity indirectly through its effect on the transportation cost parameter F and population N. A larger CBD clearly leads to more population and higher wage as stated in Corollary 1.1, and in addition, we can show that a larger CBD causes the city boundary to expand.Therefore, a larger CBD unequivocally leads to higher transportation cost F, and thus affects the rent elasticity as shown in the following corollary.

Corollary 3.1 *Residential land rent elasticity is decreasing in the CBD size if $\lambda - \sigma > (1-\xi)\beta_2 jN$; and increasing otherwise.*

See Appendix B.3 for a proof.

The last parameter that we shall explore is Λ, the fraction of residential land that is undevelopable. Like the CBD size S, Λ does not show up in Equation (34), indicating that it does not have a direct effect on land rent elasticity once we control for other exogenous and endogenous city characteristics. However, Λ does effect the population N of the city as well as its geographic size, as captured by J, and through these channels, Λ affects transportation costs and thus indirectly affects the land rent elasticity.

To understand this, consider two cities with the same population, but with different levels of Λ. The high Λ city will have a larger overall area, i.e., J, the distance from the CBD

[12] Since $F = \frac{dlog(w)}{dlog(N)}$, the term $\frac{dF}{dlog(N)}$ in χ is equivalent to $\frac{dlog(w)}{d^2log(N)}$, i.e. the second derivative of the inverse aggregate labor supply function. When F is not too large, $F \approx (\beta_1 + 2\beta_2 N)J$ is an accurate approximation as given in equation (23), thus $\frac{dF}{dlog(N)} \approx 2\beta_2 JN$ and $\chi \approx \frac{\beta_1 J + 2\beta_2 JN - 2\beta_2 JN}{2\beta_2 JN - \beta_2 jN}(1-\xi)\beta_2 jN \approx \frac{(1-\xi)\beta_1 j}{2-j/J}$. Derivation of the threshold χ is in Appendix B.1.

to the periphery will be larger. To see this, first note that for the high Λ city to have the same population, it must either have a higher productivity $\tilde{A}$ or a larger CBD, which in turn implies a higher wage, as indicated in the Corollary to Proposition 1. Next, recall that the city boundary is determined by equation (19) which shows that, given the same population, the higher wage city has a larger J. Since the high Λ city has a larger J but the same population as the unconstrained city, it has a larger transportation cost F. From the third point of Proposition 3, we conclude that cities with more undevelopable land have lower residential land rent elasticities if and only $\lambda - \sigma > (1-\xi)\beta_2 jN$

We also study the effect of undevelopable land on the city's boundary J. In Appendix B.3 we prove that the following derivative,

$$\frac{dJ}{d\Lambda} = \frac{\Lambda \frac{dN}{d\Lambda}}{(1-\xi)(\beta_1 + \beta_2 N)}[\lambda - \sigma - (1-\xi)\beta_2 JN] \tag{36}$$

is positive if and only if $\lambda - \sigma < (1-\xi)\beta_2 JN$. In other words, Λ can either increase or decrease J depending on the magnitude of other parameters. To understand this, it should be noted that population is decreasing in undevelopable residential land. The magnitude of the reduction in population depends on the agglomeration externalities; when these externalities are strong, the reduction in population is amplified. If Λ has only a modest effect on population, then the boundary will expand, since for a given boundary, there is less residential land available. However, if population declines significantly when Λ increases, the boundary can actually contract.

We summarize these properties related to undevelopable land in the following proposition.

Proposition 4 *Among cities with more undevelopable land (i.e. larger Λ)*

1. *have lower residential land rent elasticities if and only if $\lambda - \sigma > (1-\xi)\beta_2 jN$, given the same population.*

2. *have a larger geographical size if and only if $\lambda - \sigma < (1-\xi)\beta_2 JN$.*

3.3 Elasticity of Commercial Land

Substituting $A = \tilde{A}N^{\lambda}$ into the commercial bid-rent function, we have

$$p_c = \left[\frac{\tilde{A}\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}{r^{\xi}w^{1-\sigma-\xi}}\right]^{\frac{1}{\sigma}} N^{\frac{\lambda}{\sigma}} \tag{37}$$

Differentiating the above equation with respect to $\tilde{A}$, we obtain the following expression for the elasticity of commercial land rent:

$$\begin{aligned} \zeta_{p_c} &= \frac{1}{\sigma} + \frac{\lambda}{\sigma}\zeta_N - \frac{1-\sigma-\xi}{\sigma}\zeta_w \\ &= \frac{1+F}{-\lambda+\sigma+(1-\xi)F} \end{aligned} \tag{38}$$

where we have used equations (27)-(28) to substitute out ζ_w and ζ_N.

We are interested in how ζ_{p_c} changes with the agglomeration externality, transportation costs, and capital and land share in the production function. The following proposition follows from equation (38).[13]

Proposition 5 *In the benchmark model, the elasticity of commercial land rent is*

1. *increasing in λ and ξ but decreasing in σ,*

2. *decreasing in transportation cost F.*

The above proposition indicates that the elasticity of commercial land rent has similar properties as residential land, except that it always decreases with transportation cost parameter F.

As shown previously, higher transportation costs lead to higher wages, which in turn causes commercial land rent to fall. As discussed earlier, a larger share of undevelopable land, Λ, leads to higher transportation costs, F, among cities with the same population. The increase in F will in turn lower commercial land rent elasticity as indicated in Proposition 5. In addition, it is straightforward to show that a larger CBD size S always increases both the physical size and population of a city and leads to higher transportation cost F, and hence, lower commercial land rent elasticity. We summarize these results in the following corollary.

Corollary 5.1 *Commercial land rent elasticity ζ_{p_c} is*

1. *decreasing in Λ, given the same population.*

2. *decreasing in the pre-specified CBD size S.*

3.4 Discussion

As we mentioned in the introduction, we are not the first to ask how the configuration of an urban environment affects how land prices respond to shocks to either the productivity or amenities of a city. For example, Glaeser et al. (2006) and Saiz (2010) talk in terms of the elasticity of housing supply, and provide models where housing supply is more elastic will

[13]The derivative of ζ_{p_c} to F is $\frac{-\lambda-(1-\sigma-\xi)}{[-\lambda+\sigma+(1-\xi)F]^2} < 0$. Thus ζ_{p_c} always decreases with transportation cost.

respond less to shocks to demand. In this subsection we describe how our model relates to these prior contributions.

The city in our model has a number of characteristics that effectively constrain the amount of new land that becomes part of the city, and thereby effect the elasticity of supply. Since the boundary of the city in our model can expand indefinitely, there is no direct constraint; however, because workers must commute to the CBD, transportation costs effectively constrained the physical size of the city. If transportation costs are high, the effective cost of adding marginal land units is higher because new residents on the periphery will be paying higher commuting costs. As in Saiz (2012), the presence of undevelopable land is also relevant, because it determines how much new developable land is added when the boundary expands.

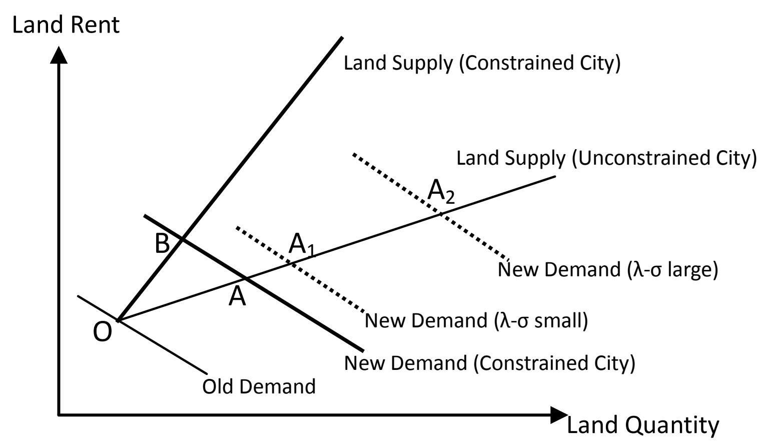
Our model illustrates that urban characteristics that increase the sensitivity of transportation costs to the growth in population decrease the sensitivity of population growth to exogenous productivity shocks, and thus dampen the extent to which agglomeration externalities amplify those shocks. As a result, and as we show in Proposition 5, the elasticity of commercial land always decreases with both increases in undevelopable land and increases in transportation costs. To understand this, first note that firms respond to an exogenous shock to productivity by hiring more workers, expanding the boundary of the city and thus increasing the commuting costs of workers at the boundary, rents in the interior, and wages. When transport costs or the portion of land that is undevelopable are higher, the effect of a productivity shock on both wages and rental rates are higher because in both cases, an increase in population results in a greater increase in the cost of commuting from the boundary. This in turn implies that the effect of the productivity shock on firm profits, and thus commercial rents, is dampened. Supply constraints also reduce the number of new workers that are hired, so the amplification effect on TFP from the agglomeration effect is also dampened.

Proposition 3 shows that effective supply constraints can have the opposite effect on residential price elasticities, but this depends on the agglomeration parameter λ. When λ is quite small, the intuition about commercial rent elasticities can be reversed. Intuitively, the residential result can be the opposite of the commercial result because the channel that makes residential rent more expensive increases wages, and thereby reduces the demand for commercial space and thus its rent. Specifically, a shock to productivity has a greater effect on rental rates when supply constraints are higher because the increased demand for workers, which expands the boundary of the city, increases commuting costs and thus rents in the interior more when transport costs are higher. This argument reflects the traditional discussion of how the elasticity of housing supply affects the sensitivity of the cost of housing to productivity shocks, e.g., Glaeser et al. (2006).

Let's now consider the case where λ is quite large. In this case, there are important feedback effects that arise from the amplification of the exogenous shock that arises because of the agglomeration externalities. Because the magnitude of this feedback effect is influenced by supply constraints, the relation between supply constraints, like transportation costs, and the price elasticity can be reversed. In particular, an exogenous shock to productivity may have less of an effect on land rents when supply is more constrained. Intuitively, this is because the constraints limit the growth in population, and thus reduce the amount by which the exogenous shock is amplified.

To summarize, supply constraints have two offsetting effects on ζ_{p_r}. The first effect, which is discussed in the existing literature, is that the constraints effectively steepen the land supply curve, causing land rent to rise more for a given demand change. The second effect, which our model illustrates, is that constraints dampen the amplification of the agglomeration effect, effectively suppressing the shift in the demand for land, causing land rent to increase less. When $\lambda - \sigma$ is sufficiently large, the second channel dominates.

Figure 2: City Constraints and Land Rent Elasticity



Note: The responsiveness of land rent to a positive productivity shock depends on both the supply elasticity of land and the strength of agglomeration effect.

We illustrate these two effects in Figure 2, which describes the supply and demand curves for land, and is similar to Figure 1 in Glaeser et al. (2006). Here constrained cities are those with more land supply constraints, represented by the steeper land supply curve. Unconstrained cites are similarly defined, represented by the flatter supply curve. Starting from the original equilibrium (point O), Glaeser et al. (2006) exposits that a positive productivity shock shifts the land demand curve to the right, crossing the land supply curve at point A for unconstrained city and point B for constrained city. Thus land rent should rise more in constrained cities. However, this analysis ignores an important channel: given the same exogenous productivity shock, the shift of demand curve also depends on whether the city is constrained or not. In constrained cities, the demand curve shifts to a less extent due to weaker agglomeration. In Figure 2, after the productivity shock, equilibrium in constrained cities are determined by the two thick lines that cross at point B. For unconstrained cities, the new demand curves are the dashed lines that cross the supply curve at point A_1 and A_2. Therefore compared with constrained cities, unconstrained cities may have lower rent ($A_1 < B$) or higher rent ($A_2 > B$), depending on the strength of the agglomeration effect.

4 Extensions

In this section we extend the benchmark model in three ways: First, in contrast to our benchmark model that had a fixed CBD, we consider an extension that allows for a flexible CBD that can expand and contract as the demand for commercial space increases or decreases. Second, in contrast to our benchmark model, that assumes that the residential part of the city can be expanded, we consider an extension that assumes that the city has a fixed boundary. Finally, in contrast to the benchmark model that assumes that capital can freely migrate in and out of the city, we consider an extension that assumes that capital in the city is immobile. With immobile capital, the price of capital, i.e. the interest rate, is endogenously determined.

Our benchmark model and its extensions consider polar extremes, and thus allow us to better understand the implications of our most important assumptions and the robustness of our main results. As we show in Appendix C, the main results about how land rent elasticity depends on city characteristics is largely true in the extended model, with a few exceptions.

The first exception is that when capital is immobile, both ζ_{p_c} and ζ_{p_r} decrease with the capital share in production ξ, which is in contrast to our earlier result that they increase with ξ in models with mobile capital. Intuitively, this is because immobile capital suppresses city expansion and contraction, and the suppressing effect is stronger when the capital share in production is larger. The second exception is that for cities with fixed boundaries, neither ζ_{p_c} nor ζ_{p_r} is affected by the transportation cost F. Intuitively, the fixed boundary itself is a form of land supply constraint, and it diminishes the effect of the implicit constraint that arises from the fact that transportation costs increase as the city expands.

Our comparison of the elasticities in the extended models with those in the benchmark model is described in the following proposition. More details about the extended models and the proof of the proposition is given in Appendix C

Proposition 6 *Relative to the benchmark model, the following is true:*

1. *fixing the city boundary,*

   (a) *residential land rent elasticity is lower if* $\lambda - \sigma > \beta_2 jN(1-\xi)$ *for all* j*,*

   (b) *commercial land rent elasticity is lower if* $\lambda - \sigma > -(1-\xi)$*.*

2. *allowing the CBD to expand and contract, land rent elasticity is higher than the benchmark model if and only if* $F < \frac{\theta}{1-\theta}$.

3. *assuming immobile capital (i.e. fixing the city-level capital stock), both commercial land and residential land have lower rent elasticities.*

As the above proposition demonstrates, the important linkage between agglomeration externalities and elasticities continues to hold. Indeed, the central message one gets from comparing the alternative models is that supply constraints described in our model can reduce or increase land rent elasticity depending on the strength of agglomeration externalities.

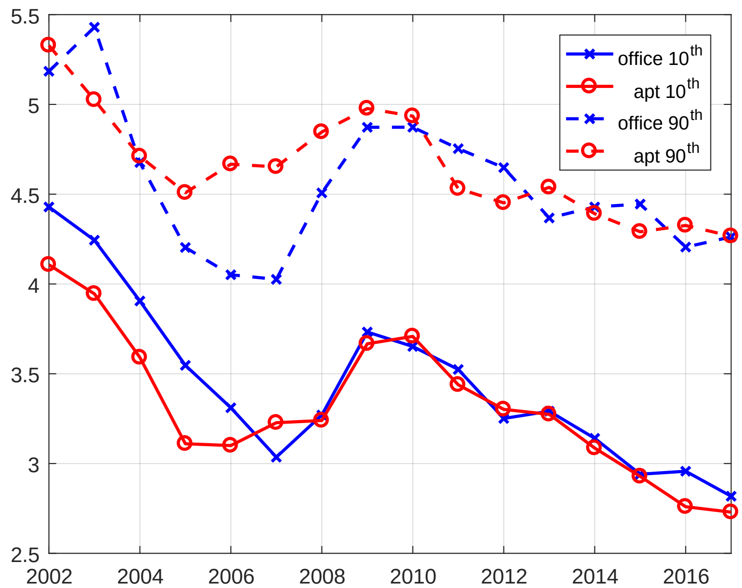
To illustrate this, consider the effect of fixing the city boundary, which is the most direct land supply constraint. Like constraints we studied earlier, the effect of this constraint depends on how large the agglomeration parameter is relative to the immobile production factor, i.e., land. When $\lambda - \sigma$ is large, the land supply constraint suppresses agglomeration externality to a larger extent and leads to smaller land rent elasticity.

Also consider the effect of having a flexible CBD, which loosens the constraints on commercial land relative to our benchmark model. As we showed in the last section, with a fixed CBD, the agglomeration externalities are offset by the fact that a growing city has less commercial space available. This constraint on commercial land lowers the marginal productivity of workers by increasing the ratio of employees to land. When the size of the CBD is flexible, commercial land expands with increases in productivity, which in turn allows the number of workers to grow more, thereby increasing the agglomeration externality. This leads to a higher residential land rent elasticity unless the city has a large transportation cost relative to the worker's preference for land, i.e. $F > \frac{\theta}{1-\theta}$ which indicates that the constraint on expansion comes mainly from the residential land market. Note that $\frac{\theta}{1-\theta}$ is much larger than 0.1 based on most of the empirical studies, so $F < \frac{\theta}{1-\theta}$ is generally true and CBD flexibility generally leads to more land rent elasticity. The flexibility of capital plays an identical role. When capital is fixed, it dampens the agglomeration externality and thus reduces the residential rent elasticity.

5 The Dynamic Model

This section explores a simple dynamic version of the static model we developed in the previous sections. The dynamic model is solved numerically, and generates rental rate volatilities, serial correlations, and rent to value ratios that can be compared to actual data on commercial rents and values from major U.S. cities. As shown in Table 5, based on data from CBRE, a large commercial broker, the standard deviation of yearly changes in office rents in the CBDs of U.S. cities average about 12% and first order serial correlations average about 27%. As shown in Figure 5, rent to value ratios reported by Real Capital Analytics, a real estate data vendor, show a large dispersion between the 10^{th} and 90^{th} percentiles for both commercial and residential real estates. Although we cannot provide formal tests on this limited data set, it is interesting to note that these variables differ across cities and that rent to value ratios appear to be more disperse for commercial properties than for multi-family.

Figure 3: Rent to Value Ratios in the Data



Note: The dispersion of rent to value ratios for apartment buildings and offices buildings. Data are from Real Capital Analytics.

Our dynamic model assumes that the exogenous productivity parameter $\tilde{A}$ follows a random walk. However, in contrast to the static model, which assumes that the benefits of agglomeration are realized immediately, the dynamic model assumes that total factor productivity in a city is a function of last year's population. We will provide various parameterizations of our model that we use to conduct simulations that allow us to gauge how quickly the model's endogenous variables, e.g., population, rents, and wages, respond to exogenous shocks, and to also measure the relation between the volatility of these parameters and various city characteristics.

Our simulations focus on two different types of cities. The first type of city is a classic monocentric city, which we call the "large city". What we have in mind here is a large city like New York, which is home to an industry, e.g., finance, that does not require substantial amounts of land and benefits a lot from agglomeration. The second type of city, which we call the "small city," houses an industry that benefits less from agglomeration and requires somewhat more land per worker. As our simulations reveal, the equilibrium population of the second type city will be substantially smaller than the first type.[14]

[14] It should be emphasized that the relevant population of a city in our model is the number of workers in the

Table 2: Volatility and Serial Correlation of Commercial Land Rent in the Data

City	Serial Corr	Volatility	City	Serial Corr	Volatility
Albuquerque	0.2913	0.0813	Miami	0.482	0.0948
Atlanta	0.3179	0.1274	Nashville	0.0967	0.0506
Austin	0.3165	0.2048	New York	0.3905	0.2198
Baltimore	0.2847	0.06	Newark	0.0599	0.095
Boston	0.3088	0.1353	Oakland	0.572	0.1013
Charlotte	0.2251	0.088	Orange County	0.1005	0.0768
Chicago	0.3386	0.0809	Orlando	0.1331	0.0746
Cincinnati	0.0342	0.1776	Philadelphia	0.4067	0.0879
Cleveland	0.18	0.2028	Phoenix	-0.0053	0.0695
Columbus	0.2877	0.2546	Portland	0.4336	0.1079
Dallas	0.3988	0.2052	Riverside	0.388	0.0806
DC	0.5857	0.138	Sacramento	0.0233	0.1117
Denver	0.434	0.1104	Salt Lake	0.4636	0.1145
Detroit	0.0896	0.0694	San Diego	0.2705	0.2658
Edison	0.1296	0.0796	San Francisco	0.2895	0.1985
Fort Lauderdale	-0.1112	0.0989	San Jose	0.1409	0.1164
Fort Worth	0.2449	0.116	Seattle	0.066	0.1172
Hartford	-0.4285	0.2361	St. Louis	0.1593	0.1213
Honolulu	0.1505	0.1088	Stamford	0.1265	0.0743
Houston	0.285	0.1278	Tampa	-0.1178	0.0804
Indianapolis	0.2079	0.1209	Trenton	0.4092	0.066
Jacksonville	0.3558	0.1296	Tucson	-0.1523	0.0651
Kansas	0.2739	0.0902	Ventura	0.2731	0.0986
LA	0.2335	0.0571	West Palm Beach	-0.3991	0.1456
Las Vegas	0.3985	0.1113	Wilmington	0.0418	0.0849
long-island	0.4451	0.1326	**Average**	**0.2143**	**0.1189**

Note: Data on rental rate of office building are from CBRE, ranging between the first quarter of 1988 and the fourth quarter of 2014. We remove the macroeconomic factors by regressing the raw data on quarter dummies. Volatility is measured as the standard deviation of logarithm.

5.1 Calibration

To study the dynamics of land rents, we modify the agglomeration effect in equation (20) as follows:

$$A_t = \tilde{A}_t N_{t-1}^{\lambda} \tag{39}$$

i.e., the agglomeration effect on productivity depends on lagged rather than contemporaneous city population.

The logarithm of the exogenous element of productivity is assumed to follow a random walk process:

$$\begin{aligned} \log \tilde{A}_t &= \log \tilde{A}_{t-1} + \epsilon_t, \\ \epsilon_t &\sim \mathcal{N}(0, \sigma_\epsilon^2) \end{aligned} \tag{40}$$

The standard deviation of the productivity shock is set to $\sigma_\epsilon = 0.003$, taken from Davis et al. (2014), and the share of consumer expenditure on land is set to $\theta = 0.3$, consistent with the estimates in Morris and Ortalo-Magne (2011). The capital share in production is $\xi = 0.2$.[15] These parameter values are listed in the top block of Table 3. In addition, we choose the values of parameters for reservation utility $\underline{u}$, agricultural rent $\underline{p}$, and the initial productivity parameter($\tilde{A}$) so that the large city has a population of 5 million and a radius of 16 kilometers. These parameter values are shown in the bottom block of Table 3.

Table 3: Parameter Values

Symbol	Definition	Value
σ_ϵ	stdev. of productivity shocks	0.003
θ	land share in preference	0.3
ξ	capital share in production	0.2
$\underline{u}$	reservation utility	0.118
$\underline{p}$	agricultural rent (per $100km^2$)	0.447
$\tilde{A}$	initial productivity	2.735

Note: This table shows the parameter values used in quantitative analysis.

For the large city, the parameter of agglomeration is set to $\lambda = 0.08$, which is about the upper-bound bound in Ahlfeldt et al. (2015).[16] The share of land in production is $\sigma = 0.05$.

city's CBD. In reality large metropolitan areas, like Dallas-Fort Worth, contain a number of different CBDs, e.g., downtown Dallas and Fort Worth and Plano. Within the context of our model Dallas-Fort Worth should be viewed as three (or more) medium size cities rather than one megalopolis. We will leave a more thorough analysis of the poly-centric cities for future research.

[15] This is in line with the estimates for service and investment industry in Valentinyi and Herrendorf (2008).

[16] The Ahlfeldt et al. (2015) estimates come from a structural model that exploits the exogenous variation from the division and reunification of Berlin.

The share of undevelopable land is initially set to $\Lambda = 0$ and the transportation cost function is given by equation (41), which represents the costs in a car-based city. We assume that there is no fixed component in a car based city, but the cost of traveling by car increases with distance, and because of road congestion, transportation costs increase for any distance as population grows.

$$
\begin{aligned}
f(j, N|car) &= 0.0000 + 0.0018 \times j + 1.50e^{-9} \times j \times N & (41) \\
f(j, N|rail) &= 0.0285 + 0.0017 \times j + 1.15e^{-9} \times j \times N. & (42)
\end{aligned}
$$

The simulations also consider a rail-based transportation technology with a cost function given by equation (42). We choose the function parameters that generate the same population for the rail-based city as its car-based counterpart with the same CBD size and city boundary. Note that the rail-based transportation cost function has a large fixed component but exhibits smaller increases with both distance and population.

Given $\sigma = 0.05$ and $\xi = 0.2$, the implied labor share in production is $1 - \sigma - \xi = 0.75$, which is consistent with the idea that the large city houses an industry in which human capital plays a large role, and it is higher than the average labor share world-wide documented in Karabarbounis and Neiman (2014).

5.2 Initial City Configuration

We start by describing the relation between our parameter values and the steady state size and prices in the cities we study. Using the parameter values given in Table 3 and the transportation cost functions shown in equation (41), we obtain the city configuration as shown in the first row of Table 4. Note that although the population, CBD size and radius are effectively engineered by our choices of $\underline{u}$, $\underline{p}$ and $\tilde{A}$, given these parameters, the model generates wage, commercial land rent and residential land rents to clear the labor and land markets. We also report residential density, defined as residents per 100 square meters of residential land in the city.

The second row of Table 4 shows how the city configuration changes when 40% of the available residential land cannot be developed (the pre-specified CBD size, reservation utility and agricultural land rent are held constant). Relative to the city where 100% of the residential land can be developed, the population of the city with undevelopable land declines from 5 million to 3.88 million, and the city radius expands from 16 kilometers to 18.28 kilometers. It is noteworthy that the reduction in the availability of residential land causes residential rents to fall slightly, which illustrates our result that when agglomeration externalities are large, constraints on developable land does not necessarily increase prices. Commercial land rent falls quite a bit with the introduction of residential supply constraints and wages also decline because workers are less productive in smaller cities. It should also be noted that because the more constrained city has a larger radius, residential density actually falls.

Table 4: Initial City Configuration

	Population (*million*)	*CBD* (km^2)	*Radius* (*km*)	Wage	p_c ($per100m^2$)	p_r ($per100m^2$)	*Density* ($pop/100m^2$)
λ=0.08, σ=0.05	5.00	30	16.00	3.14	3.49	0.73	62.20
$\Lambda = 0.4$	3.88	30	18.28	3.11	2.68	0.71	61.66
λ=0.076, σ=0.05	2.50	30	11.70	2.88	1.60	0.56	58.06
λ=0.076, σ=0.15	1.72	30	9.79	2.82	3.73	0.52	57.06
$\Lambda = 0.4$	1.58	30	12.09	2.84	3.46	0.53	57.41

Note: Initial city configurations. p_c is the commercial land rent in the CBD, and p_r is the residential land rent next to the CBD. $\Lambda = 0.4$ denotes the case where 40% of the residential land is undevelopable. Density is the average population density of residential land, calculated as total population divided by total developable residential land.

The third row of Table 4 considers a city that houses an industry with lower agglomeration externalities. As we show in Table 4, a small change in λ makes a large difference. Lowering λ from 0.08 to 0.076 and keeping all other parameters the same, the city population is reduced by 50%. A snowballing effect is evident: lower λ leads to lower productivity, and hence lower population, which further reduces productivity.

The fourth row of Table 4 describes a smaller city that is home to an industry that utilizes more land in its production function. Specifically, we increase the land parameter, σ, from $\sigma = 0.05$ to $\sigma = 0.15$. The larger land share implies smaller labor share in production, which implies the city population is further reduced, which in turn reduces the city radius. In this city, residential land rent is lower, but because of the increased importance of land in production, commercial land rent rises considerably. It should be noted that we are considering a case where the physical size of the CBD is fixed. In a model where the CBD can grow, the physical size of the CBD could be larger and commercial rents could be lower.

The last row in Table 4 illustrates how the small city changes when 40% of its land is undevelopable land. As was the case for the large city, the undevelopable land causes population to fall but radius to expand. In contrast to the larger city that hosts firms that require less commercial land, the constraint on the supply of residential causes residential land rent to rise. As we illustrated in Figure 2, this is because the effect from reduced agglomeration is dominated by the effect of less available residential land. Commercial land rent and population density fall, but to a less extent compared with the large city.

It should be noted that the individuals in all the cities described in Table 4 enjoy the same level of utility and the firms in these cities are endowed with the same exogenous productivity shocks. Yet we see significant cross-city differences in wages and even larger difference in land rents. This illustrates how cross-city differences in city characteristics, as well as different productivity shocks, can generate the cross-city house price differences studied in Gyourko et al. (2013) and Nieuwerburgh and Weill (2010).

5.3 Volatility

To simulate possible paths the economy may take, we draw one hundred sample paths of exogenous productivity shocks based on the stochastic process shown in equation (40), each path consists of one hundred periods (years). Then we feed each exogenous productivity path into the economy and calculate the resulting endogenous elements of the economy. Using these simulated economies, we calculate the average volatilities for each urban configuration. The volatility measure we use is the standard deviation of the logarithm of rents.

5.3.1 Large Cities

Results for the large cities are reported in the first two rows of Table 5. The average volatility of the exogenous productivity shocks in our simulated samples is 1.122. These shocks are amplified to create a volatility of total factor productivity of 1.849 in the car-based city and 2.161 in the rail-based city. Volatilities of all the endogenous variables are larger if the city is rail-based, for the reasons illustrated in Figure 2. We are particularly interested in the volatility of commercial land rent, which is about 11% in a car-based city and 15% in a rail-based city, which is roughly comparable to the average volatility of 12.19% for office rent, which is reported in Table 5.

Table 5: Volatility (std. of logarithm)

λ=0.08, σ=0.05	*Productivity* (*endogenous*)	*Wage*	*Population*	p_c	p_r j=0	p_r j=5
Baseline (Car)	1.849	1.715	9.564	11.276	5.715	4.780
Rail	2.161	1.847	13.712	15.552	6.158	5.180
$\Lambda = 0.4$ (car)	1.869	1.694	10.286	11.977	5.647	4.861
Fix capital (car)	1.354	0.577	3.110	3.686	1.922	1.610
Fix boundary (car)	1.341	1.493	2.944	4.437	4.976	4.679
λ=0.076, σ=0.05						
Baseline (car)	2.707	2.059	21.348	23.387	6.863	5.586
λ=0.076, σ=0.15						
Baseline (car)	1.840	0.469	9.765	10.233	1.564	1.219
Rail	1.935	0.349	11.044	11.392	1.163	0.908
$\Lambda = 0.4$ (car)	1.805	0.512	9.307	9.818	1.708	1.406
Fix capital (car)	1.333	0.300	2.952	3.252	0.999	0.813
Fix boundary (car)	1.293	1.165	2.405	3.570	3.882	3.712

Note: Volatility is measured as the standard deviation of the logarithm. p_c and p_r denote commercial land rent and residential land rent, respectively. $j = 0$ and $j = 5$ denote residential locations next to the CBD and 5 kilometers from the CBD, respectively. $\Lambda = 0.4$ denotes the case where 40% of the land in each residential location is undevelopable.

In the third row of the table we consider the case where 40% of the land is undevelopable.

As we showed in the previous table, the city has a lower population and is less dense in this case. As shown in this table, the presence of undevelopable land increases the volatility of commercial property, it slightly lowers the volatility of residential property close to the CBD but slightly increases the volatility of residential property further from the CBD. The modest changes in volatility reflect two opposing forces. On one hand, undevelopable land leads to a smaller population and hence reduces congestion, which causes volatility to rise. On the other hand, undevelopable land constrains supply, which dampens volatility when λ is large relative to σ as we discussed earlier. Thus the net effect of undevelopable land on land rent volatility is small.

If capital is immobile, then volatilities are much smaller, as shown in the row labeled "Fix capital". Volatilities are also lower when the city has a fixed boundary, which is consistent with our theoretical prediction that land supply constraints leads to lower land rent elasticity when the agglomeration effect is strong.

5.3.2 Small Cities

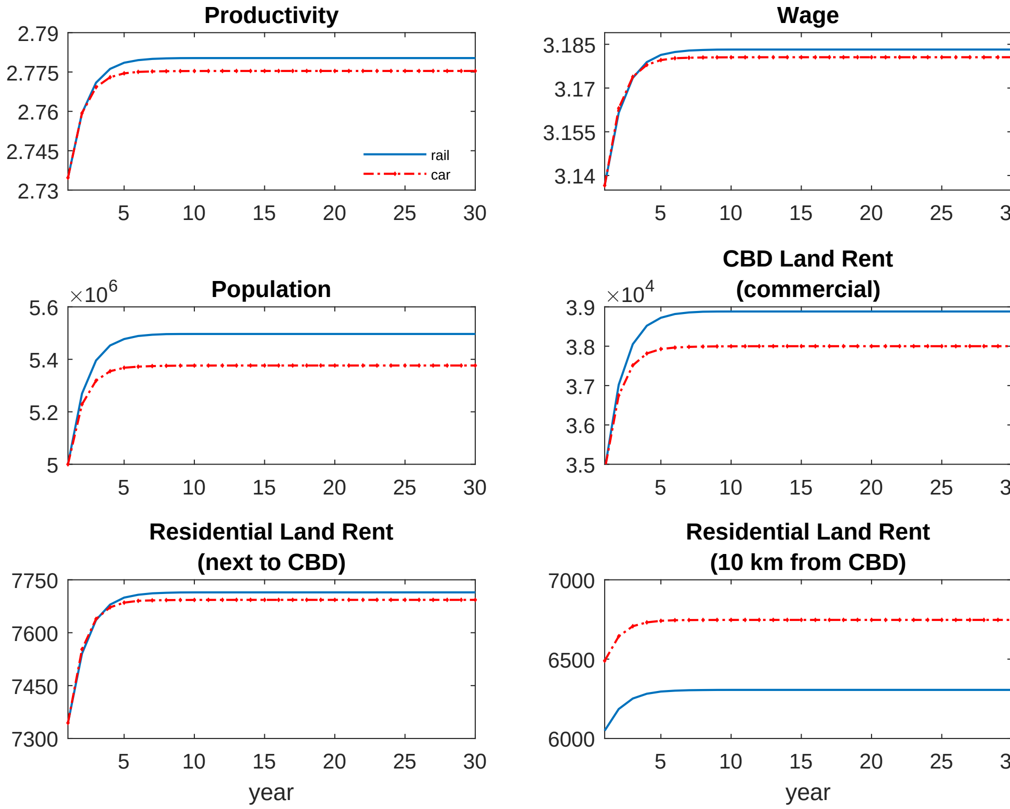
As shown in the middle block of Table 5, the city with slightly lower agglomeration externalities is substantially more volatile. As mentioned earlier, the smaller agglomeration parameter leads to very unstable cities, unless land share in production is increased. Therefore we focus on the low agglomeration city with $\sigma = 0.15$. The corresponding volatilities is reported in the bottom block of Table 5. Clearly, compared with the large city, volatilities of wage and residential land rent are lower almost by a factor of 4. However, the volatility of commercial land rent is reduced only slightly, which is partly driven by the larger land share in production.

Note that when $\lambda = 0.076$ and $\sigma = 0.15$, $\lambda - \sigma$ is negative, which means that land supply constraints should increase residential land rent volatilities as discussed earlier. This is illustrated in the last block of Table 5. Compared with car-based transportation, the rail-based transportation is associated with significantly lower residential rent volatilities because land supply is less constrained with rail transportation. With undevelopable land and fixed city boundaries, the city is effectively more constrained, and residential land rent is clearly more volatile.

5.4 Serial Correlation

As we show in Table 5 the growth rate of commercial property rents exhibit positive serial correlation that tends to differ from city to city. The growth rate of office rents in major US cities has an average serial correlation of 27.31, but differs substantially from city to city. In our model, given our assumption that the agglomeration effect is based on lagged population in the city, the rise and fall of a city due to productivity shocks are gradual. As a result, the

Figure 4: Transition



growth of land rent exhibits persistence even though the productivity shock itself is assumed to follow a random walk.

Figure 4 plots the transition of the city economy when the city receives a shock of three standard deviations of the exogenous productivity. Although the shock affects exogenous productivity only in the current period, the response of each endogenous variable is persistent because of the agglomeration effect. The top left panel, which presents the transition of endogenous productivity, shows how feedback from the agglomeration externality leads to persistent changes in total factor productivity. As the figure shows, the responses of endogenous variables are larger and more persistent in the benchmark city with a rail transport system. Everything else equal, a car based city has a smaller and less persistent response.

Using the same sample paths described earlier, we calculate the serial correlations of the growth rate of land rent. As shown in Table 6, the serial correlation varies considerably with city characteristics. A comparison of the top block with the bottom block of the table reveals that the serial correlation is lower in the small city. On average, the serial correlation from our model is in line with the serial correlation reported in Table 5.

It is interesting to note that land supply constraints, which do not always generate higher volatilities, generate lower serial correlations in all of the configurations we consider. Intuitively,

Table 6: Serial Correlation

λ=0.08, σ=0.05	p_c	p_r (j=0)	p_r (j=5)
Baseline	0.366	0.356	0.357
Rail	0.455	0.432	0.433
Λ=0.4	0.391	0.380	0.381
Fix capital	0.139	0.139	0.139
Fix boundary	0.131	0.131	0.132
λ=0.076, σ=0.15			
Baseline	0.357	0.351	0.351
Rail	0.386	0.380	0.379
Λ=0.4	0.345	0.342	0.342
Fix capital	0.126	0.125	0.125
Fix boundary	0.100	0.100	0.100

Note: Cities are car-based unless stated otherwise. p_c and p_r denote commercial land rent and residential land rent, respectively. $j = 0$ and $j = 5$ denote residential locations next to the CBD and 5 kilometers from the CBD, respectively. $\Lambda = 0.4$ denotes the case where 40% of the land in each residential location is undevelopable.

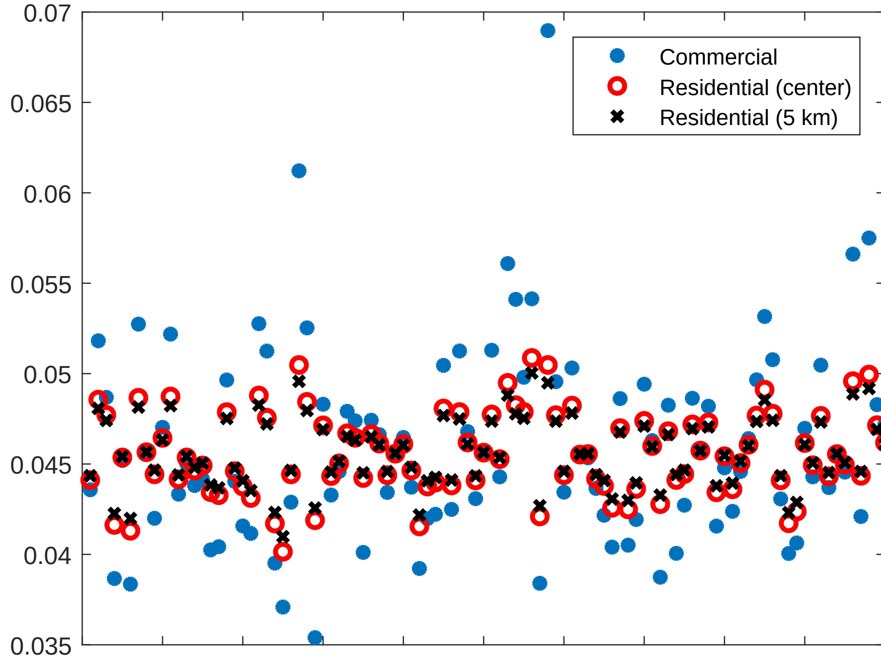
land supply constraints dampen serial correlations by reducing the amount that population responds to an exogenous shock to productivity. If population responds less, the subsequent increase in total factor productivity is also less. Having a fixed residential boundary has an especially significant effect on serial correlations as well as volatilities: it lowers the persistence of residential land rent in location $j = 0$ from 0.351 to 0.100, but raises the volatility from 1.564 to 3.882.

5.5 Rent to Value Ratio

Rent to value ratios reflects expectations about future rent increases. In this subsection we explore the extent to which our model can generate the cross-city dispersion in rent to value ratios observed in the data. To explore the dispersion in rent to value ratios we simulate our model, for each set of exogenous parameters, by drawing 100 year sample paths of exogenous productivity $\tilde{A}$ as described in subsection 5.3. In the simulations, cities of the same type are initially identical, but they become increasingly different over time due to the different realizations of productivity shocks.

We calculate the distributions of rent to value ratios in year 50, dividing the realized year 50 rents by values calculated as the discounted sums of expected rental income from years 51 through 100. The discount rate is assumed to be 4% per year, thus the land rent to value ratio is about 0.04 in a steady state.[17]

Figure 5: Rent to Value Ratios



Note: Given the initial configuration as shown in the first row of Table 4, we simulation the city economy for 100 years. This figure shows the distribution of land rent to value ratio in year 50. Note the cities are car-based and the agglomeration effect is strong.

Figure 5 shows the scatter plot of rent to value ratios in year 50 of simulation for large cities from the benchmark model. The dispersion of ratios is clearly larger for commercial land – the minimum and maximum of the rent to value ratio are 0.035 and 0.069, respectively. For residential land next to the CBD, the difference between the minimum ratio and maximum ratio is about 0.01, and the difference is somewhat larger for locations close to the CBD.

To summarize the distribution shown in Figure 5, we rank cities by their rent to value ratios and report the ratio of the 10^{th} percentile city and that of the 90^{th} percentile city. Table 7 shows these statistics. As the first row shows, from the benchmark model the rent to value ratios of commercial land are 4.01% and 5.27% for the 10^{th} and 90^{th} percentiles respectively. This amounts to a difference of 1.26% which is 27.32% of the average rent to value ratio, as reported in the block labeled "Dispersion" in the table. The dispersions are 13.54% and 4.7% of the mean rent to value ratio for residential land next to the CBD and 5 kilometers away from the CBD, respectively.

[17]Numerically we can only sum up the rent stream for finite years, thus the land rent to value ratio is slightly larger than 0.04 in a steady state. Note that cities have smaller rent to value ratios if they are small in population in year 50, or if they receive large positive productivity shocks more often between year 50-100, or both.

Table 7: Land Rent to Value Ratio (%)

	10^{th} **percentile**			90^{th} **percentile**			**Dispersion** $100\times(90^{th}-10^{th})/mean$		
λ**=0.08,** σ**=0.05**	L_c	L_r (j=0)	L_r (j=5)	L_c	L_r (j=0)	L_r (j=5)	L_c	L_r (j=0)	L_r (j=5)
Baseline	4.01	4.25	4.60	5.27	4.87	4.83	27.32	13.54	4.70
Rail	3.85	4.24	4.60	5.72	4.92	4.86	39.18	14.90	5.41
Λ=0.4	3.91	4.24	4.61	5.40	4.88	4.84	31.98	13.98	4.89
Fix capital	4.36	4.46	4.58	4.74	4.65	4.66	8.24	4.28	1.76
Fix boundary	4.32	4.29	4.59	4.77	4.79	4.84	9.89	11.09	5.16
λ**=0.076,** σ**=0.15**									
Baseline	4.01	4.48	4.58	5.11	4.64	4.63	24.14	3.49	1.14
Rail	3.95	4.50	4.57	5.17	4.62	4.62	26.75	2.60	0.89
$\Lambda = 0.4$	4.04	4.47	4.58	5.09	4.65	4.64	23.08	3.85	1.25
Fix capital	4.39	4.51	4.57	4.72	4.61	4.61	7.25	2.22	0.89
Fix boundary	4.36	4.35	4.59	4.74	4.74	4.78	8.31	8.54	4.19

Note: Land rent to value ratios in term of percentages. Cities are car-based unless stated otherwise. For each type of city, given the initial configuration, we simulate the city economy for 100 years. We rank cities based on their land rent to value ratio in year 50, then we reports the average rent to value ratios for the 10^{th} and 90^{th} percentiles. Cities that collapse are dropped in the calculation.

Among cities with large agglomeration effects, the dispersions between the 10^{th} percentile and the 90^{th} percentile are larger in rail-based cities than in car-based cities. When cities have undevelopable residential land (i.e. the case of $\Lambda = 0.4$), the dispersion is also larger relative to the baseline case. This is because cities with undevelopable land have smaller populations, which allow them to rise and fall to a larger extent. The dispersion in rent to value ratios are also fairly small in cities with fixed boundaries and fixed capital, since these constraints limit the amount by which the cities can grow. Table 7 also shows that the dispersion in rent to value ratios are smaller for the smaller cities with lower agglomeration externalities. In particular, among small agglomeration cities, rail-based cities have less disperse rent to value ratios than car-based cities, which is the opposite of the pattern observed in large agglomeration cities. This is consistent with our theory that transportation cost, as a form of land supply constraint, leads to more response of residential land rent to productivity shocks when the agglomeration effect is weak.

Using a sample of 54 major US cities between 2002-2017, we find that the dispersion between the 10^{th} percentile and the 90^{th} percentile is 27.95% of the mean value for office buildings, and the dispersion is 33.17% for apartments. Thus our model does generate the larger dispersion in the rent to value ratios of commercial buildings observed in the data, aside from the extreme cases of fixed capital and fixed city boundary. But our model generates much smaller dispersion in the rent to value ratios of residential land than in the data.

The large dispersion in the rent to value ratio of residential land in the data is likely to be caused by some missing ingredients in our simple model. For example, our model does not include consumption externalities, which have been shown to be important.[18] Another missing ingredients in our model is the heterogeneity of workers in terms of skills or human capital levels. As Gyourko et al. (2013) and other studies show, high human capital workers tend to sort themselves into superstar cities like New York and San Francisco. Further, these rich households select to cluster in prime locations of a city (gentrification) as studied in Guerrieri et al. (2013). Our conjecture is, with the inclusion of heterogeneity workers, the model would generate larger dispersions in rent to value ratios of residential land, both between cities and within cities.

6 Conclusion

Although the financial crisis had a number of causes, an important contributor was the perception that real estate is a relatively low risk investment. This misperception created an overly levered property sector as well as overly exposed financial institutions, some of which failed.

The model developed in this paper provides a framework for thinking about how the design of a city and the firms that inhabit it affect the size of the city, its land values and the risk of its real estate. We start by extending the seminal monocentric urban models in ways that allow us to more carefully consider how effective constraints on urban growth, like the presence of undevelopable land and congestion, interact with industrial characteristics, like agglomeration externalities and the role of land in the firms' production functions. This static model is then extended to a dynamic model that we solve numerically. By simulating shocks to the exogenous element of a city's productivity, we are able to explore how the structure of the city and its industrial base affects the volatility of its population and land rents.

Before one takes our numerical analysis too seriously, it should be noted that we made a number of assumptions to improve tractability, which probably should be relaxed in future work. Most importantly, the model does not actually include physical buildings, which are both slow to build and slow to depreciate. Including buildings to our model is likely to dampen the volatility of land rents if it takes time to construct new buildings. Intuitively, part of benefit of a productivity shock will be captured by increases in the value of structures that are temporarily in short supply, which means that less will be captured by increases in the raw land.[19] Adding

[18] Glaeser et al. (2001) is one of the earlier papers that point to the role of cities as centers for consumption as well as production. Rossi-Hansberg et al. (2010) and Ahlfeldt et al. (2015) estimate how residential externalities increase with residential density.

[19] Glaeser and Gyourko (2005) make the point that the slow depreciation of urban buildings slows the decline of cities that experience negative productivity shocks.

building structures to a dynamic model like ours is clearly warranted, but it is beyond the scope of the current paper.[20]

It should also be noted that although productivity shocks are the sole source of uncertainty in the model we simulate, our model suggests a number of variables that have a meaningful influence on urban land values, and many of these are also likely to be stochastic. For example, we show that the agglomeration externality has a large influence on both population and land rents. It is likely that in some urban areas with human capital based industries, like the Bay Area, these agglomeration benefits have increased, increasing both population and land rents. However, improvements in telecommunication technologies and other innovations can potentially reduce the relative benefits of physical proximity, thereby lowering agglomeration externalities and reducing property values. Uncertainty about these technologies is clearly an important source of real estate risk.

Likewise, our model points to transportation technology as an important source of uncertainty. Indeed, increases in total factor productivity, either from exogenous shocks or from increases in the benefits of agglomeration, lead to meaningful increases in population and real estate prices only if the effects are not dampened by transportation costs. Hence, improvements in rail transit and autonomous driving technologies can potentially have a substantial effect on both prices and the evolution of uncertainty.

Finally, we should note that since undevelopable land plays an important role in our model, uncertainty about future zoning decisions is a source of risk. In our model, zoning choices that increase the amount of residential land that can be developed always increases the value of commercial land. However, residential land can increase or decrease in value depending on its location, agglomeration externalities, and congestion effects. Given the conflicting interests of the various parties, political outcomes that determine future property values are likely to be difficult to predict.

[20]Landowners, in such a model have an incentive to wait to build, it might be interesting to develop a model where the density of the city increases as uncertainty is resolved.

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Appendices

A Equilibrium in the Benchmark Model

The benchmark model has seven endogenous variables $\{p_r, p_c, w, N, K, J, A\}$ that are determined by seven equations: (13), (14), (15), (16), (17), (18), and (20). In this appendix we show that the seven equation can be reduced into two equations: the aggregate labor supply equation and aggregate labor demand equation. Then we prove Proposition 1

A.1 Aggregate Labor Supply/Demand Equations

Aggregate Labor Supply Since we assume each location has $1-\Lambda$ unit of land, the clearance of land market in location j is given by

$$n(j)h(j) = 1 - \Lambda,$$

thus the number of workers residing in location j.

$$n(j) = \frac{1-\Lambda}{h(j)} = \frac{(1-\Lambda)p_r(j)}{\theta w e^{-f(j,N)}}$$

The total number of workers residing in the city is the integral of workers in each location, as shown in equation (17). Now we substitute out land rent using the residential bid-rent function to obtain:

$$\begin{aligned} N &= \int_{j=0}^{J} \frac{(1-\Lambda)p_r(j)}{\theta w e^{-f(j,N)}} dj \\ &= \frac{(1-\Lambda)B_0}{\theta} w^{\frac{1-\theta}{\theta}} \int_0^J e^{-\frac{1-\theta}{\theta} f(j,N)} dj \end{aligned} \tag{43}$$

Taking logarithm of the above equation leads to the aggregate labor supply equation which is equation (22).

Given the transportation cost function $f(j,N) = \beta_0 + \beta_1 j + \beta_2 jN$, the transportation gradient is $\frac{\partial f(j,N)}{\partial j} = \beta_1 + \beta_2 N$, thus the term $\int_0^J e^{-\frac{1-\theta}{\theta} f(j,N)} dj$ can be re-written as

$$\begin{aligned} \int_0^J e^{-\frac{1-\theta}{\theta} f(j,N)} dj &= \int_0^J \frac{1}{-\frac{1-\theta}{\theta}\frac{\partial f(j,N)}{\partial j}} de^{-\frac{1-\theta}{\theta} f(j,N)} \\ &= -\frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)} \int_0^J de^{-\frac{1-\theta}{\theta} f(j,N)} \\ &= -\frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)} \left(e^{-\frac{1-\theta}{\theta} f(J,N)} - e^{-\frac{1-\theta}{\theta} f(0,N)} \right) \\ &= \frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)} \left(e^{-\frac{1-\theta}{\theta} \beta_0} - e^{-\frac{1-\theta}{\theta} f(J,N)} \right) \end{aligned} \tag{44}$$

where we have used $f(0,N) = \beta_0$.

The city boundary J is endogenously determined by equating its rent with the exogenous agricultural rent, i.e. $p_r(J) = \underline{p}$, which is equivalent to $B_0[we^{-f(J,N)}]^{1/\theta} = \underline{p}$ using the bid-rent function. That is

$$f(J,N) = log(w) - \theta \left(log \frac{\underline{p}}{B_0} \right) \tag{45}$$

Substituting this boundary condition into equation (44), we obtain:

$$\int_0^J e^{-\frac{1-\theta}{\theta} f(j,N)} dj = \frac{\theta}{(\beta_1+\beta_2 N)(1-\theta)} \left(e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta}\left[log(w) - \theta\left(log\frac{\underline{p}}{B_0}\right)\right]} \right) \tag{46}$$

With the above equation, the aggregate labor supply function (equation 22) can be rewritten into:

$$\begin{aligned} log(N) &= log\left[\frac{(1-\Lambda)B_0}{1-\theta}\right] + \frac{1-\theta}{\theta} log(w) - log(\beta_1 + \beta_2 N) \\ &\quad + log\left[e^{-\frac{1-\theta}{\theta}\beta_0} - e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}\right] \end{aligned} \tag{47}$$

Aggregate Labor Demand In equations (15), total labor input relative to land is determined by commercial land rent relative to wage. Using the commercial bid-rent function to substitute out land rent, we obtain:

$$\begin{aligned} \frac{N}{S} &= \frac{1-\sigma-\xi}{\sigma}\frac{p_c}{w} \\ &= \left[\frac{A\xi^{\xi}(1-\sigma-\xi)^{1-\xi}}{r^{\xi}w^{1-\xi}}\right]^{\frac{1}{\sigma}} \end{aligned} \tag{48}$$

To take the agglomeration into account, we to substitute out A with $A = \tilde{A}N^{\lambda}$ (equation 20), and re-write equation (48) as

$$N = \left[\frac{r^{\xi}w^{1-\xi}}{\tilde{A}\xi^{\xi}(1-\sigma-\xi)^{1-\xi}S^{\sigma}}\right]^{\frac{1}{\lambda-\sigma}}, \tag{49}$$

Taking logarithm of the above equation leads to equation (21), the aggregate labor demand function.

Solving Other Variables From $\{w, N\}$ Once we find the market clearing $\{w, N\}$, we can solve for p_r, p_c, K, J and A from the residential bid-rent function, commercial bid-rent function, equation (16), equation (45), and equation (48) respectively. ∎

A.2 Proof of Proposition 1

To prove the proposition, first we show the slope of aggregate labor supply curve is close to infinity when wage and population are small, implying that transportation costs are near zero, and a small increase in the wage rate causes a large in-migration of workers. The slope converges to $\frac{1-\theta}{\theta}$ when wage and population keep growing. Next, we show the aggregate labor demand curve may have two crossings with the aggregate labor supply curve.

Slope of Aggregate Labor Supply Curve From equation (47), the derivative of $log(N)$ with respect to $log(w)$ is:

$$\frac{dlog(N)}{dlog(w)} = -\frac{\beta_2 N}{\beta_1 + \beta_2 N} \times \frac{dlog(N)}{dlog(w)} + \frac{1-\theta}{\theta}\left(1 + \frac{e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}}{e^{-\frac{1-\theta}{\theta}f(0,N)} - e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}}\right).$$

Therefore

$$
\begin{aligned}
\frac{dlog(N)}{dlog(w)} &= \left(\frac{\beta_1+\beta_2 N}{\beta_1+2\beta_2 N}\right)\left(\frac{1-\theta}{\theta}\right)\left(\frac{e^{-\frac{1-\theta}{\theta}f(0,N)}}{e^{-\frac{1-\theta}{\theta}f(0,N)}-e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}}\right) \\
&= \left(\frac{\beta_1+\beta_2 N}{\beta_1+2\beta_2 N}\right)\left(\frac{1-\theta}{\theta}\right)\left(\frac{e^{-\frac{1-\theta}{\theta}f(0,N)}}{e^{-\frac{1-\theta}{\theta}f(0,N)}-e^{-\frac{1-\theta}{\theta}f(J,N)}}\right) \qquad (50) \\
&= \frac{1}{F}
\end{aligned}
$$

where the definition of F is given in equation (24). Here we have used $f(J,N) = log(w) - \theta\left(log\frac{p}{B_0}\right)$, which is equation (45). Equation (50) indicates:

1. When $log(N)$ and $log(w)$ are small, distance from the CBD to the boundary J is near zero, thus the slope given by equation (50) converges to infinity as the term $e^{-\frac{1-\theta}{\theta}f(J,N)}$ converges to $e^{-\frac{1-\theta}{\theta}f(0,N)}$, and $\frac{\beta_1+\beta_2 N}{\beta_1+2\beta_2 N}$ converges to one.

2. When $log(N)$ and $log(w)$ approaches infinity, the slope given by equation (50) converges to $\frac{1-\theta}{2\theta}$ because the term $e^{-\frac{1-\theta}{\theta}f(J,N)}$ converges to zero and the term $\frac{\beta_1+\beta_2 N}{\beta_1+2\beta_2 N}$ converges to 2 as N converges to infinity.

In conclusion, the aggregate labor supply function is an increasing function that is concave downward.

Aggregate Labor Demand Curve and the Number of Equilibrium (Equilibria) The slope of aggregate labor demand curve, as given in equation (21), is $\frac{1-\xi}{\lambda-\sigma}$. When $\lambda < \sigma$, $\frac{1-\xi}{\lambda-\sigma} < 0$ and the curve is downward sloping. Clearly the curve crosses the aggregate labor supply curve once, and the equilibrium is unique.

If $\lambda > \sigma$, then the aggregate labor demand curve is upward sloping. It crosses the aggregate labor supply curve at least once since the latter has a near-infinity slope when wage and population are small. If λ is not large, then the slope $\frac{1-\xi}{\lambda-\sigma}$ is larger than $\frac{2\theta}{1-\theta}$ which is the slope of aggregate labor demand curve when wage and population are large. In this case, the two curve will cross twice, leading to two equilibria.

Thus the necessary and sufficient condition for the existence of two equilibria is that the aggregate labor demand curve is steeper than the aggregate labor supply curve when wage and population converge to infinity, i.e. $\frac{1-\xi}{\lambda-\sigma} > \frac{2\theta}{1-\theta}$, which is equivalent to $\sigma < \lambda < \sigma + (1-\xi)\frac{2\theta}{1-\theta}$.

Finally, if

$$\lambda > \sigma + (1-\xi)\frac{2\theta}{1-\theta},$$

then the aggregate labor demand curve is flatter than the aggregate labor supply curve, and the city keeps expanding with population and wage converging to infinity. This is the case we rule out in the paper. ∎

A.3 Elasticity of City Boundary to Productivity

First, we derive the elasticity of ζ_J as given in equation (30). Using equation (19) and taking the derivative of J with respect to $log(\tilde{A})$, we obtain

$$\begin{aligned}
\frac{dJ}{dlog(\tilde{A})} &= \frac{\frac{dlog(w)}{dlog(\tilde{A})}(\beta_1+\beta_2 N)+\beta_2\frac{dN}{dlog(\tilde{A})}(\beta_1+\beta_2 N)J}{(\beta_1+\beta_2 N)^2} \\
&= \frac{\frac{dlog(w)}{dlog(\tilde{A})}+\beta_2\frac{dlog(N)}{dlog(\tilde{A})}JN}{\beta_1+\beta_2 N} \\
&= \frac{\zeta_w+\zeta_N\beta_2 JN}{\beta_1+\beta_2 N}
\end{aligned}$$

Therefore,

$$\zeta_J = \frac{dlog(J)}{dlog(\tilde{A})} = \frac{\zeta_w-\beta_2 JN\zeta_N}{(\beta_1+\beta_2 N)J} = \frac{F-\beta_2 JN}{(\beta_1+\beta_2 N)J}\zeta_N$$

where we have used equation (27) to substitute out ζ_w.

Next, we show that $F-\beta_2 JN>0$ if the following is true:

$$(\beta_1+2\beta_2 N)J<\frac{2\theta}{1-\theta}$$

which is condition (32) in the main body of the paper. From Taylor expansion, it is true that $1-e^{-x}>x-\frac{x^2}{2}$ for $x>0$, thus we have the following:

$$\begin{aligned}
F &= \frac{\theta}{1-\theta}\left(1-e^{-\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J}\right)\left(\frac{\beta_1+2\beta_2 N}{\beta_1+\beta_2 N}\right) \\
&> \frac{\theta}{1-\theta}\left[\frac{1-\theta}{\theta}(\beta_1+\beta_2 N)J-\frac{1}{2}\left(\frac{1-\theta}{\theta}\right)^2(\beta_1+\beta_2 N)^2 J^2\right]\left(\frac{\beta_1+2\beta_2 N}{\beta_1+\beta_2 N}\right) \\
&= (\beta_1+2\beta_2 N)J-\frac{1-\theta}{2\theta}(\beta_1+\beta_2 N)(\beta_1+2\beta_2 N)J^2 \\
&> (\beta_1+2\beta_2 N)J-\frac{1}{(\beta_1+2\beta_2 N)J}(\beta_1+\beta_2 N)(\beta_1+2\beta_2 N)J^2 \\
&= \beta_1 JN
\end{aligned} \tag{51}$$

where we have used $-\frac{1-\theta}{2\theta}>-\frac{1}{(\beta_1+2\beta_2 N)J}$ which follows from condition (32). Thus condition (32) implies $F-\beta_2 JN>0$. ∎

A.4 Residential Land Rent Elasticity When Congestion Effect Is Not Location-Specific

Here we prove equation (35) which is the residential land rent elasticity when the the transportation cost function is $f=\beta_0+\beta_1 j+\beta_2 N$. That is, if the congestion effect is not location-specific, then residential land rent elasticity is not location-specific either.

With the new transportation cost function, we have $\frac{\partial f(j,N)}{\partial j} = \beta_1$, thus we rewrite equation (44) as the following:

$$\int_0^J e^{-\frac{1-\theta}{\theta}f(j,N)}dj = \frac{\theta}{(1-\theta)\beta_1}\left[e^{-\frac{1-\theta}{\theta}(\beta_0+\beta_2 N)} - e^{-\frac{1-\theta}{\theta}f(J,N)}\right]$$

Substituting this into equation (43), the aggregate supply equation becomes:

$$\begin{aligned} log(N) &= log\left[\frac{(1-\Lambda)B_0}{(1-\theta)\beta_1}\right] + \frac{1-\theta}{\theta}log(w) \\ &\quad +log\left[e^{-\frac{1-\theta}{\theta}(\beta_0+\beta_2 N)} - e^{-\frac{1-\theta}{\theta}\left[log(w)-\theta log\left(\frac{p}{B_0}\right)\right]}\right] \end{aligned}$$

Differentiating both sides of the above equation with respect to $log(w)$, we obtain the following:

$$\frac{dlog(N)}{dlog(w)} = \frac{1-\theta}{\theta} + \frac{-\frac{(1-\theta)\beta_2 N}{\theta}e^{-\frac{1-\theta}{\theta}f(0,N)}\frac{dlog(N)}{dlog(w)} + \frac{1-\theta}{\theta}e^{-\frac{1-\theta}{\theta}f(J,N)}}{e^{-\frac{1-\theta}{\theta}f(0,N)} - e^{-\frac{1-\theta}{\theta}f(J,N)}},$$

which leads to the following slope of the aggregate labor supply curve:

$$\frac{dlog(N)}{dlog(w)} = \frac{\frac{1-\theta}{\theta}}{1+\frac{1-\theta}{\theta}\beta_2 N - e^{-\frac{1-\theta}{\theta}\beta_1 J}} := \frac{1}{F^\star}. \tag{52}$$

Here $F^\star$ is similarly defined as F for the case of location-specific congestion in the benchmark model. From the above equation, the expression for $F^\star$ is

$$F^\star = \frac{\theta}{1-\theta}\left(1-e^{-\frac{1-\theta}{\theta}\beta_1 J}\right) + \beta_2 N \approx \beta_1 J + \beta_2 N \tag{53}$$

Using equation (52), it is easy to show the following:

$$\frac{\zeta_w}{\zeta_N} = \frac{dlog(w)/dlog(\tilde{A})}{dlog(N)/dlog(\tilde{A})} = \frac{dlog(w)}{dlog(N)} = F^\star, \tag{54}$$

which is the similar to equation (27) for the case of location-specific congestion. Note that the aggregate labor demand equation is not affected by the new transportation cost function, thus we substitute equation (54) into equation (28) to obtain:

$$\zeta_N = \frac{1}{-\lambda+\sigma+(1-\xi)F^\star} \tag{55}$$

With the new transportation cost function, $\frac{df(j,N)}{dlog(\tilde{A})} = \beta_2 N\zeta_N$. Thus we obtain the following by differentiating the residential bid-rent function with respect to $log(\tilde{A})$:

$$\zeta_{p_r^\star} = \frac{1}{\theta}(\zeta_w - \beta_2 N\zeta_N) \tag{56}$$

Equations (54)-(56) lead to equation (35). ∎

B Dependence of Residential Land Rent Elasticity on Population, CBD Size and Undevelopable Land

This appendix shows how residential land rent elasticity, as given in equation (34), depends on the city population N, the CBD size S, and the share of undevelopable land Λ.

B.1 City Population

Here we show that the item about how rent elasticity depends on city population in **Proposition 3** is true. From equation (34), the derivative of residential land elasticity to city population is:

$$
\begin{aligned}
\frac{d\zeta_{p_r}}{dN} &= \frac{\left(\frac{dF}{dN} - \beta_2 j\right)[-\lambda + \sigma + (1-\xi)F] - \frac{dF}{dN}(1-\xi)(F - \beta_2 jN)}{\theta[-\lambda + \sigma + (1-\xi)F]^2} \\
&= \frac{(-\lambda + \sigma)\left[\frac{dF}{dN} - \beta_2 j\right] + (1-\xi)\beta_2 j\left[\frac{dF}{dN}N - F\right]}{\theta[-\lambda + \sigma + (1-\xi)F]^2} \\
&= \frac{(-\lambda + \sigma)\left[\frac{dF}{dlog(N)} - \beta_2 jN\right] + (1-\xi)\beta_2 jN\left[\frac{dF}{dlog(N)} - F\right]}{\theta N[-\lambda + \sigma + (1-\xi)F]^2}
\end{aligned} \tag{57}
$$

Recall that F is the slope of the inverse aggregate labor supply curve, and $\lim_{N\to\infty} F = \frac{2\theta}{1-\theta}$, i.e. the slope becomes a constant. Thus as N tends toward infinity, we have $\frac{dF}{dlog(N)} \to 0$. Therefore,

$$
\lim_{N\to\infty} \frac{d\zeta_{p_r}}{dN} = \frac{\beta_2 j[\lambda - \sigma - (1-\xi)F]}{\theta[-\lambda + \sigma + (1-\xi)F]^2} < 0,
$$

since $\lambda - \sigma < (1-\xi)F$ from regularity condition (26). More generally, when N is large such that $\frac{dF}{dlog(N)} \leq \beta_2 jN$, we show that $\frac{d\zeta_{p_r}}{dN} < 0$. To see this, first note that $\frac{dF}{dlog(N)} \leq \beta_2 jN$ implies $\frac{dF}{dlog(N)} < F$ since F is larger than $\beta_2 jN$, thus the two bracketed terms in the numerator of equation (57) are both non-positive, i.e., $\left[\frac{dF}{dlog(N)} - \beta_2 jN\right] \leq 0$ and $\left[\frac{dF}{dlog(N)} - F\right] < 0$. In the case of $-\lambda + \sigma \geq 0$, so clearly $\frac{d\zeta_{p_r}}{dN} < 0$. In the case of $-\lambda + \sigma < 0$, using the regularity condition of $\lambda - \sigma < (1-\xi)F$, the numerator of equation (57) becomes

$$
\begin{aligned}
& (-\lambda + \sigma)\left[\frac{dF}{dlog(N)} - \beta_2 jN\right] + (1-\xi)\beta_2 jN\left[\frac{dF}{dlog(N)} - F\right] \\
= \;& (\lambda - \sigma)\left[\beta_2 jN - \frac{dF}{dlog(N)}\right] - (1-\xi)\beta_2 jN\left[F - \frac{dF}{dlog(N)}\right] \\
< \;& (1-\xi)F\left[\beta_2 jN - \frac{dF}{dlog(N)}\right] - (1-\xi)\beta_2 jN\left[F - \frac{dF}{dlog(N)}\right] \\
= \;& -(1-\xi)\frac{dF}{dlog(N)}(F - \beta_2 jN) \\
< \;& 0,
\end{aligned}
$$

hence

$$\frac{d\zeta_{p_r}}{dN} < 0 \qquad when \quad \frac{dF}{dlog(N)} \leq \beta_2 jN$$

i.e., as the city population gets exceedingly large, more population always leads to smaller land rent elasticity.

We focus on the case of $\frac{dF}{dlog(N)} > \beta_2 jN$, i.e., $\frac{dF}{dlog(N)} - \beta_2 jN > 0$. From equation (57), $\frac{d\zeta_{p_r}}{dN} < 0$ is equivalent to

$$(-\lambda + \sigma)\left[\frac{dF}{dlog(N)} - \beta_2 jN\right] < (1-\xi)\beta_2 jN\left[F - \frac{dF}{dlog(N)}\right]$$

Since $\left[\frac{dF}{dlog(N)} - \beta_2 jN\right]$, this is equivalent to

$$\lambda - \sigma > -\left[\frac{F - \frac{dF}{dlog(N)}}{\frac{dF}{dlog(N)} - \beta_2 jN}\right](1-\xi)\beta_2 jN := -\chi$$

That is, $\frac{d\zeta_{p_r}}{dN} < 0$ if and only if $\lambda - \sigma > -\chi$. ▮

B.2 CBD Size

B.3 Undevelopable Land

Proof Here we show how the city boundary J changes with S, the size of pre-specified CBD. We also prove equation (36) which shows how J changes with Λ, the share of undevelopable land.

C More Details on Extended Models

This Appendix provides more details about the extended model and proves Proposition 6.

C.1 Expandable CBD

In this subsection, we maintain the monocentric city assumption, but allow the CBD size to be determined by competition between firms and residents for the land at the CBD border. In this setting, an increase in the exogenous productivity parameter increases the demand for commercial land, which in turn leads to an increase in the geographic size of the CBD.

C.1.1 Equilibrium CBD Size

Land use competition ensures the equality of residential rent and commercial rent on the border of the CBD, i.e. $p_{r(j=0)} = p_c$. Substitute out $p_{r=0}$ and p_c using the residential and commercial bid rent functions (equations 13-14) and taking the agglomeration effect into account, this equality implies the following:

$$log(S) = \frac{1}{\lambda} log\left(\frac{r^{\xi}(B_0 e^{-\beta_0/\theta})^{\sigma-\lambda}}{\tilde{A}\xi^{\xi}\sigma^{\sigma-\lambda}(1-\sigma-\xi)^{1-\sigma-\xi+\lambda}}\right) + \frac{1-\theta}{\lambda\theta}\left[\sigma - \lambda + \frac{\theta}{1-\theta}(1-\xi)\right] log(w), \quad (58)$$

Using (58), we can substitute for S in the aggregate labor demand function (equation 21) to obtain the following aggregate labor demand equation:

$$log(N) = \frac{1}{\lambda} log\left(\frac{r^{\xi}(B_0 e^{-\beta_0/\theta})^{\sigma}}{\tilde{A}\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}\right) + \frac{1}{\lambda}\left(\frac{\sigma}{\theta} + 1 - \sigma - \xi\right) log(w). \quad (59)$$

Clearly the new aggregate labor demand function is upward sloping unless $\lambda = 0$. Therefore with $\lambda > 0$, this extended model always has two equilibria. In contrast, two equilibria arise in the benchmark model only when the agglomeration effect is strong enough, i.e. $\lambda > \sigma$. Intuitively, this is because flexible CBD strengthens the agglomeration.

The extended model here shares the same aggregate labor supply function as the benchmark model (i.e. equation 22), since the function is derived from the partial equilibrium in the residential land market, and it is not affected by the expandable CBD. As in the benchmark model, the aggregate labor demand function and aggregate labor supply function determine the equilibrium wage and population, which in turn determines the other endogenous variables.

As in the benchmark model, the city grows explosively if the slope of aggregate labor demand curve is flatter than that of aggregate supply curve. We rule this out by imposing the condition $\lambda < 2\sigma + (1-\xi)\frac{2\theta}{1-\theta}$.

C.1.2 Elasticities

Differentiating equation (59) with respect to $log(\tilde{A})$, we obtain obtain:

$$\zeta_N = \frac{1}{\lambda}\left(\frac{\sigma}{\theta} + 1 - \sigma - \xi\right)\zeta_w - \frac{1}{\lambda}. \quad (60)$$

This, combined with equation (27), yields the following population elasticity:

$$\zeta_N = \frac{1}{\left(\frac{\sigma}{\theta} + 1 - \sigma - \xi\right) F - \lambda} \quad (61)$$

This population elasticity shares some properties as in the benchmark model. It is increasing in λ and ξ, but decreasing in the land share σ. As in the benchmark model, we rule out the unstable small city equilibrium by imposing the condition $\zeta_N > 0$.

Based on equation (27), (33), and (61), the elasticity of land rent is:

$$\zeta_{p_r(j)} = \frac{1}{\theta} \times \frac{F - \beta_2 j N}{\left(\frac{\sigma}{\theta} + 1 - \sigma - \xi\right) F - \lambda} \quad (62)$$

Compared with the benchmark model, the elasticity in equation (62) is larger if and only if $F < \frac{\theta}{1-\theta}$. Thus everything else equal, land rent is more responsive to a productivity shock if transportation cost is small relative to the worker's preference for land. Since $\frac{\theta}{1-\theta}$ is much larger than 0.1 based on most of the empirical studies, we take $F < \frac{\theta}{1-\theta}$ as given in what follows.

We state important properties for cities with expandable CBDs in following proposition.

Proposition 7 *When the CBD is flexible, land rent elasticity $\zeta_{p_r(j)}$ is*

1. *larger than land rent elasticity in the benchmark model in each location provided that $F < \frac{\theta}{1-\theta}$.*
2. *increasing in λ and ξ but decreasing in σ.*
3. *decreasing in distance to the CBD.*
4. *decreasing in F if $\lambda > \left(\frac{\sigma}{\theta} + 1 - \sigma - \xi\right)\beta_2 jN$; but increasing otherwise.*

C.2 Fixed boundary

This subsection considers cities where the city boundary is fixed rather than endogenous. This new assumption affects the aggregate labor supply function since J in equation (22) is now exogenous, thus differentiating the equation with respect to $\tilde{A}$ yields:

$$\begin{aligned} \frac{\zeta_w}{\zeta_N} &= \frac{\theta}{1-\theta}\frac{\beta_1 + 2\beta_2 N}{\beta_1 + \beta_2 N} - \frac{\beta_2 JN}{e^{-\frac{1-\theta}{\theta}(\beta_1 J + \beta_2 JN)} - 1} \\ &\approx \frac{\theta}{1-\theta}, \end{aligned} \tag{63}$$

Thus in cities with fixed boundaries, the ratio $\frac{\zeta_w}{\zeta_N}$ mainly depends on the worker's preference for land, while in expandable cities the ratio mainly depends on transportation cost as shown in equation (27).

We have derived equation (28) from the aggregate labor demand function in the benchmark model. Combining this with equation (63) to substitute out ζ_w and ζ_N in (33), we derive the following expression for elasticity of land rent in a city with fixed boundaries

$$\zeta_{p_r} = \frac{1}{\theta} \times \frac{\frac{\theta}{1-\theta} - \beta_2 jN}{-\lambda + \sigma + (1-\xi)\frac{\theta}{1-\theta}} \tag{64}$$

Using equation (37), the elasticity of commercial land rent is

$$\zeta_{p_c} = \frac{\frac{1}{1-\theta}}{-\lambda + \sigma + (1-\xi)\frac{\theta}{1-\theta}}, \tag{65}$$

which indicates that commercial land rent elasticity is independent of transportation cost when the city has a fixed boundary. Thus given a productivity shock, the land demand effect of poor transportation (i.e. less demand shift) exactly cancels the land supply effect.

We summarize important comparative statics in the following proposition.

Proposition 8 *For cities with fixed boundaries*

1. *For both residential land and commercial land, rent elasticity is increasing in λ and ξ, decreasing in σ (same as in the benchmark model).*

2. *Residential land rent elasticity is decreasing in distance to the CBD (same as in the benchmark model).*

3. *Commercial land rent elasticity is not affected by transportation cost (different from the benchmark model).*

4. *Compared with the benchmark model:*

   (a) *residential land rent has lower elasticity if $\lambda - \sigma > \beta_2 jN(1-\xi)$ for all j,*

   (b) *commercial land rent has lower elasticity if $\lambda - \sigma > -(1-\xi)$.*

where for point (4) we have imposed the condition $F < \frac{\theta}{1-\theta}$.

Fixed city boundaries are clearly a form of land supply constraint. As illustrated earlier, the supply constraint is represented a steeper supply curve, causing land rent to response more to a productivity shock. However the supply constraint also dampens the agglomeration effect in production, causing land demand curve to shift less in response to the same productivity shock. Point (4) of the proposition again indicates that when the agglomeration effect is strong enough, cities with fixed boundaries have lower land rent elasticity. In addition, for residential land rent, the positive production externality represented by $\lambda - \sigma$ needs to dominate the negative congestion externality as represented by $\beta_2 jN(1-\xi)$.

C.3 Immobile Capital

Thus far we have assumed that capital can flow into and out of the city at zero cost. In this subsection, we consider the alternative assumption of immobile capital. This is partly motivated by the observations in Glaeser and Gyourko (2005) that the depreciation of urban buildings are slow which causes the slow decline of cities that experience negative productivity shocks.

C.3.1 Endogenous Capital Price

We assume the city has a fixed $\bar{K}$ stock of capital, and the price of capital r is endogenously determined by the capital market clearing condition. From the firm's problem we show that $\frac{k}{n} = \frac{\xi}{1-\sigma-\xi}\frac{w}{r}$ (equation 11) for each firm. Aggregating over all the firm we have:

$$r = \frac{\xi}{1-\sigma-\xi}\frac{N}{\bar{K}}w \tag{66}$$

The rental rate of capital rises with productivity. Given the higher productivity, both wage and total number of workers rise, but $\bar{K}$ stays the same, thus equation (66) predicts that r should rise. In other words, due to immobility capital owners share part of the economic benefits (costs) from the rising (falling) productivity.

C.3.2 Elasticities with Pre-specified CBD

With this endogenous rental rate of capital, we shall substitute out r in the aggregate labor demand function. For the case of pre-specified CBD, we rewrite equation (21) as:

$$log(N) = \frac{1}{\lambda-\sigma-\xi}log\left(\frac{1}{\tilde{A}(1-\sigma-\xi)\bar{K}^{\xi}S^{\sigma}}\right) + \frac{1}{\lambda-\sigma-\xi}log(w) \tag{67}$$

Differentiating equation (67) with respect to $\tilde{A}$, we have the following:

$$\zeta_N = \frac{1-\zeta_w}{\sigma+\xi-\lambda} \tag{68}$$

Residential Land: The aggregate labor supply function from the benchmark model is not affected by capital immobility assumption, because it is derived from the partial equilibrium in the residential land market. Therefore, equations (27) still holds. Together with equations (68), it substitutes out ζ_w and ζ_N in equation (33) to reach:

$$\zeta_{p_r(j)} \;\; = \;\; \frac{1}{\theta} \times \frac{F-\beta_2 jN}{-\lambda+\sigma+\xi+F} \tag{69}$$

This equation indicated that, when capital is immobile, land rent elasticity decreases with ξ, the share of capital in production. This is the opposite of the result in the benchmark model. The contrast is intuitive: when the capital is immobile, a larger ξ imposes more suppression on production externality, thus less response of land rent to productivity shocks. Compared with equation (34), it is clear that land rent is less elastic to productivity shocks relative to the benchmark model.

We have the following proposition:

Proposition 9 *If capital is immobile, for cities with pre-specified CBD, residential land rent elasticity is:*

1. *smaller than cities with perfect capital mobility;*

2. *increasing in λ, and decreasing in σ, ξ, and distance from the CBD;*

3. *decreasing in transportation cost if $\lambda > \sigma + \xi + \beta_2 jN$, and decreasing otherwise*

Commercial Land: Using equation (66), we substitute out r in commercial bid-rent function (equation 37) to obtain:

$$p_c = \left[\frac{\tilde{A}\sigma^{\sigma}(1-\sigma-\xi)^{1-\sigma}\bar{K}^{\xi}}{w^{1-\sigma}}\right]^{\frac{1}{\sigma}} N^{\frac{\lambda-\xi}{\sigma}}$$

Thus the elasticity of commercial land rent is

$$\begin{aligned}\zeta_{p_c} &= \frac{1}{\sigma} + \frac{\lambda-\xi}{\sigma}\zeta_N - \frac{1-\sigma}{\sigma}\zeta_w \\ &= \frac{1+F}{-\lambda+\sigma+\xi+F} \end{aligned} \tag{70}$$

Compared with equation (38) in the benchmark model, it is clear that commercial land rent has lower elasticity when capital is immobile. We summarize important properties of ζ_{p_c} in the following proposition.

Proposition 10 *If capital is immobile, for cities with pre-specified CBD, the elasticity of commercial land rent ζ_{p_c} is:*

1. *increasing in λ, but decreasing in σ, ξ, and transportation cost;*

2. *independent of the CBD size S and the share of unusable residential land Λ.*

3. *smaller than cities with perfect capital mobility and CBD segmentation;*

C.3.3 Elasticities with Expandable CBD

For the case of flexible CBD segmentation, we rewrite equation (59) as:

$$log(N) = \frac{1}{\lambda-\xi} log\left(\frac{(B_0 e^{-\beta_0/\theta})^{\sigma}}{\tilde{A}\sigma^{\sigma}\xi^{\xi}(1-\sigma-\xi)^{1-\sigma-\xi}}\right) + \frac{1}{\lambda-\xi}\left(\frac{\sigma}{\theta}+1-\sigma\right) log(w). \tag{71}$$

Differentiating equation (71) with respect to $\tilde{A}$, we have

$$\zeta_N = -\frac{1}{\lambda-\xi} + \frac{1}{\lambda-\xi}\left(\frac{\sigma}{\theta}+1-\sigma\right)\zeta_w \tag{72}$$

Substituting this and equation (27) into (33), we derive the following expression related to the elasticity of land rent when the CBD is expandable:

$$\zeta_{p_r(j)} = \frac{1}{\theta} \times \frac{F - \beta_2 jN}{\left(\frac{\sigma}{\theta} + 1 - \sigma\right) F + \xi - \lambda} \tag{73}$$

Compared with equation (62) which is for cities with perfect capital mobility, obviously land rent has lower elasticity here. The following proposition describes the main properties of this land rent elasticity.

Proposition 11 *If capital is immobile, for cities with flexible CBDs, the elasticity of land rent is*

1. *smaller than a city with perfect capital mobility;*
2. *increasing in λ, and decreasing in σ, ξ, and distance from the CBD;*
3. *decreasing in F if $\lambda > \xi + \left(\frac{\sigma}{\theta} + 1 - \sigma\right) \beta_2 jN$; but increasing otherwise.*