How Should Central Bank Issue Digital Currency?

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Abstract

This paper develops a micro-founded general equilibrium model of payments to investigate the optimal design of a retail CBDC along three policy dimensions: interest payment, holding limit, and fee or subsidy on use. It highlights the role of a holding limit in addressing disintermediation concerns, the substitutability of policy tools in achieving welfare objectives, and the potential for balancing CBDC adoption with deposit retention by subsidizing households' CBDC use when a minimum holding requirement is met.

1 Introduction

With advancements in mobile payment technology, the growth of online retailing, and shocks such as the COVID-19 pandemic, the decline in cash use by the public has become both inevitable and pronounced. Although electronic payment methods can be more efficient than traditional paper money in the sense that they can be used for online and large transactions, the declining use of central bank money can hinder critical central bank objectives, such as promoting financial inclusion and effectively implementing monetary policies. These concerns, along with other considerations such as providing the public with a safe and private digital central bank money, have prompted central bankers to consider issuing retail central bank digital currencies (CBDCs), including European central bank (ECB), bank of England (BoE), and People's Bank of China (PBoC). While issuing retail CBDCs can help central banks regain their influence, it may also lead to undesirable consequences, with disintermediating private banking being one of the most prominent concerns. A retail CBDC can potentially crowd out bank deposits by raising the funding costs for commercial banks. To mitigate such potential adverse effects, the design of the retail CBDC must be approached with careful consideration. One design feature proposed by central bankers to help retain bank deposits is the imposition of a cap on the amount of CBDC that households can hold. The ECB has considered a €3,000 limit per individual for digital euro holdings, while the BoE has suggested a cap between £10,000 and £20,000. Meanwhile, the PBoC has introduced multiple limits for the digital yuan, including a single payment limit, a daily cumulative limit, and a balance cap, which vary depending on whether the digital yuan is anonymous or linked to an individual's identity.

This paper studies how the design of a CBDC along three dimensions affects deposit creation, investment, and welfare in an environment where CBDC competes with cash and private bank deposits as a medium of exchange. The three design features of CBDC that central bankers can manipulate for their policy objectives are the interest payment, a holding limit, and a fee or subsidy on CBDC use. Specifically, this paper aims to answer the following questions. Will a holding limit on individuals' CBDC balances help mitigate the disintermediation effect? Can the disintermediation concern be addressed through alternative policy tools, such as interest payment on CBDC balances, or the imposition of fees or subsidies on CBDC use, and how might these tools influence households' portfolio decisions? How should central banks use policy tools at their disposal to achieve different goals? When a CBDC is introduced, can it coexist with existing forms of money such as cash and bank deposits, and if so, what form would this coexistence take? If a central bank were to offer subsidies to encourage CBDC adoption, would it necessarily lead to more disintermediation? This paper addresses these questions by developing a model of banking and means of payment, featuring two perfectly segmented markets, each with distinct payment methods available. Cash and bank deposits can only be used in offline and online markets respectively while CBDC, if introduced, can be used in both markets. Households in offline and online markets make their portfolio decisions based on different characteristics of available assets for exchange. Therefore, the model allows for an investigation of the effects of different CBDC designs on the existing public money and private banks. The model shows that what might seem a novel instrument may not be, and what may sound like a straightforward result could not be.

I show that when the holding limit on CBDC is binding, in equilibrium, CBDC can coexist with cash and deposits either at the extensive margin or the intensive margin, depending on its design, where there is extensive margin coexistence of two means of payment when some households accumulate one medium of exchange and others accumulate the other, and there is intensive margin coexistence of two means of payment when households hold both media of exchange in their portfolios. Moreover, in both cases, by setting a lower holding limit on CBDC, bank deposits are less crowded out. In the equilibrium where deposits coexist with CBDC at the intensive margin, the interest payment and the holding limit on CBDC can be viewed as a single policy tool in terms of their impact on welfare, whereas the CBDC fee becomes ineffective as the welfare function in this equilibrium is independent of it. In contrast, in the equilibrium where deposits coexist with CBDC at the extensive margin, the welfare impact of the interest payment and the holding limit on CBDC are not exactly the same, and the CBDC fee switches to an effective tool. The model also shows that when focusing solely on the offline market, the holding limit on CBDC influence households' portfolio decisions in a linear fashion. As the holding limit increases, offline households first prefer the cash-only portfolio, then the mixed portfolio including both cash and CBDC, and finally the CBDC-only portfolio.

A holding limit on CBDC can be introduced to mitigate its disintermediation effect. However, to make CBDC issue effective in practice, central banks would still prefer households to hold larger amounts of CBDC if they choose to adopt it. Can this policy goal be achieved while minimizing the impact on bank deposits? The model demonstrates that this can be accomplished by introducing an additional policy tool: a minimum CBDC holding requirement to qualify for a reduced CBDC fee or even a subsidy. As this minimum threshold increases, the equilibrium deposit rate declines, and bank deposits are subsequently crowded in rather than crowded out. Meanwhile, households who choose to accumulate CBDC will hold the higher minimum threshold accordingly.

My work is related to several lines of literature on CBDCs. First, this paper complements research investigating the effects of new monetary policy tools from introducing a CBDC such as Barrdear and Kumhof (2022), Davoodalhosseini (2022), Hua and Zhu (2021) and Brunnermeier and Niepelt (2019). Second, this paper contributes to the strand of literature that studies the impact of a retail CBDC on the private banking sector and welfare (e.g., Chiu et al. (2023);Chiu and Davoodalhosseini (2023);Keister and Sanches (2022);Williamson (2022);Andolfatto (2021)). Some of these papers also discuss different designs of a CBDC, but they primarily focused on the interest payment and the types of transactions in which CBDC can be used. Williamson (2022) also examines the differing welfare impacts of issuing CBDC through existing bank deposit contracts versus a narrow banking facility. Agur et al. (2022) studies the optimal level of anonymity that a CBDC should provide in the presence of network effects, both when it is interest-bearing and when it is not. Similarly, Wang (2020) studies the optimal design of CBDC in the context of tax evasion. Building on and different from these works, this paper also examines other design features of a CBDC, analyzing the effects of a holding limit, as well as a fee or subsidy on CBDC use, alongside the interest payment, on private intermediaries and overall welfare.

This paper also establishes a theoretical basis for existing empirical literature and discussions documenting the appropriate size of the cap on CBDC holdings. For example, Li et al. (2024) shows that a large holding limit of 25,000 Canadian digital dollars could effectively prevent disruptions to the financial system. Bidder et al. (2024) shows that issuing a retail CBDC with a holding limit is welfare-improving and suggests an optimal holding limit of €1,500 for digital euro.

The rest of the paper is organized as follows. Section 2 builds the model. Section 3 and section 4 characterize equilibrium without CBDC and with CBDC respectively. Section 5 conducts welfare analysis. Section 6 concludes.

2 The model

The model is based on the frameworks of Lagos and Wright (2005) and Keister and Sanches (2022). Time is discrete and with infinite horizon. Each period is divided into two subperiods: a frictional decentralized market (DM) and a Walrasian centralized market (CM).

2.1 Agents

There are four types of agents: a unit measure each of buyers and sellers, a continuum of bankers, and the central bank. Buyers and sellers first meet in the DM, then they enter the CM. There are two perishable goods which are produced and consumed in the two subperiods respectively: a DM good and a CM good.

Buyers and sellers live forever and the discount factor across periods is $\beta \in (0,1)$. In the DM, buyers and sellers meet bilaterally, where buyers consume what sellers produce and not vice versa. By consuming q units of DM good, a buyer's utility is $u(q)$ with $u'(0) = \infty$, $u' > 0$, and $u'' < 0$. Sellers incur a linear cost in producing q units of DM good. Hence, the efficient amount of DM trade maximizing the total surplus, denoted as q^*, solves $u'(q^*) = 1$. In the CM, both buyers and sellers can work and consume the CM good. Agents can produce one unit of CM good with unitary labor input, which generates one unit of disutility. Their preference for CM consumption is $U(\cdot)$ with $U'(0)=\infty$, $U'>0$, and $U''<0$. In summary, buyers' and sellers' period utilities are

$$U^b(q,X,h) = u(q) + U(X) - h,$$
$$U^s(q,X,h) = -q + U(X) - h,$$

where X is consumption of the CM good and h is labor input.

Bankers of a given generation are born in current CM. They become old and die in the next CM, and only consume when they are old. A new cohort of bankers is born each CM. Young bankers are endowed with an investment project but they have no internal funds to finance. Therefore, they must borrow from households by issuing deposits with interest payment. Investment projects take one unit of current CM good as input and have heterogeneous returns in the next CM, as in Keister and Sanches (2022). Bankers' project returns are uniformly distributed in the support $[0,\bar{\gamma}]$, with a measure η of bankers at each point. Banks' project returns are not fully pledgeable. Young bankers can only pledge up to a fraction ε of their project returns to depositors. Moreover, banks are subject to a reserve requirement that they can only invest a fraction $1-\mu$ of issued deposits into projects. The rest goes into reserves at the central bank.

The central bank issues three types of liabilities: cash, reserves, and CBDC if introduced. Cash and CBDC are liquid in that they can be used to facilitate exchanges, whereas reserves are illiquid.

2.2 Assets and exchange

Agents other than the central bank lack commitment and there is no record-keeping technology among buyers and sellers so DM trade must be quid pro quo. I use take-it-or-leave-it (TIOLI) offers made by buyers as the trading protocol in the DM. Buyers will have to use a means of payment in order to consume in the DM. Furthermore, households are perfectly and permanently [1]segmented into two markets where different means of payment can be used for transactions before they make their portfolio choices: a fraction λ_1 of households engages in the offline market, while the remaining fraction $\lambda_2 \equiv 1-\lambda_1$ participates in the online market. Before CBDC is introduced, households can only use cash and deposits in the offline and online market respectively. When CBDC is introduced, it is assumed that CBDC can be used in both markets. The three possible media of exchange are embedded with different characteristics. First, they have different rates of return. Returns on cash and CBDC are controlled by the central bank through targeting the inflation rate and interest payment on CBDC. The deposit rate is determined in equilibrium which either clears the deposit market or makes households indifferent between accumulating deposits and CBDC. Second, there is a cap on the amount of CBDC that buyers can choose to hold, which is denoted as x. Third, there are fixed costs associated with using deposits and CBDC borne by buyers, which are summarized as $f \equiv f_1 + f_2$ and $\delta \equiv \delta_1 + \delta_2$ respectively. The fixed costs have two components. f_1 and δ_1 are user costs while f_2 and δ_2 are fees collected by banks. For the economy as a whole, the former represent true costs, while the latter are merely transfer costs. User costs

[1]Results are the same when the segmentation is random each period.

include resources, both human and physical ones, invested in adopting the underlying technology that supports a specific payment method. Aside from it, there are fees paid for maintaining bank accounts at private banks and the central bank. In the case of CBDC, δ_2 can take on positive values, which means the central bank subsidizes households' use of CBDC. Moreover, I assume that the overall cost of using CBDC, δ, is strictly positive given that it is still costly for the public to starting using a new payment method in general.

The flow of payments and goods is as follows. In the CM, buyers in offline and online markets make their portfolio choices as which means of payment to bring into the next DM for trade knowing their types. When bank deposits and CBDC are accumulated, fixed costs are payable at this stage. In the following DM, trade takes place where terms of trade is determined by TIOLI offers made by buyers. In the next CM, sellers redeem deposits received with old bankers and consume. Old bankers consume their project returns after paying back deposits and interest. Buyers receive lump-sum transfers and adjust their balances. Figure 1 summarizes activities carried out by private agents in a timeline.

2.3 Asset demand

In this section, I solve buyers' portfolio problems in both offline and online markets within a generalized setup to derive households' asset demand for the three possible means of payment: cash, deposits and CBDC.

In what follows, time subscripts on period-t variables are omitted, while variables from $t-1$ are labeled with subscript -1, and variables from $t+1$ are labeled with subscript $+1$. Define $\vec{a} = (c,d,e)$ as the portfolio vector of real balances of cash, deposits and CBDC accumulated by a buyer. Denote the price of money in terms of the CM good in period t as ϕ, in period $t-1$ as ϕ_{-1}. The net nominal interest payment on CBDC balances is i. Let $\vec{R} = (R^c, R^d, R^e)$ be the vector of real gross returns provided by cash, deposits and CBDC, where $R^c = \frac{\phi}{\phi_{-1}}$, $R^e = \frac{(1+i)\phi}{\phi_{-1}}$, and R^d is determined in equilibrium. Let W_s and V_s denote buyers' CM and DM value functions in market $s \in \{1,2\}$, where 1 refers to the offline market and 2 refers to the online market. In the CM, buyers in offline and online markets choose their consumption of the CM good X^b, labor h, and portfolio $\vec{a}_{+1} = (c_{+1}, d_{+1}, e_{+1})$ carried into the next DM.

$$
\begin{aligned}
W_s(\vec{a}) = \max_{X^b, h, \vec{a}_{+1}} \quad & U(X^b) - h + \beta V_s(\vec{a}_{+1}) \\
\text{subject to} \quad & X^b + \vec{1}\cdot\vec{a}_{+1} = h + \vec{R}\cdot\vec{a} + T - \mathbb{1}_{\{d_{+1}>0\}} \times f - \mathbb{1}_{\{e_{+1}>0\}} \times \delta, \\
& e_{+1} \leq x,
\end{aligned}
$$

where $\vec{1} = (1,1,1)$, "$\cdot$" is the inner product, and T is the lump-sum transfers received by buyers in real terms. f and δ are fixed costs incurred by holding deposits and CBDC.

Assuming an interior solution for h and substituting h from the budget constraint, the CM value function can be rewritten as:

$$
W_s(\vec{a}) = R\cdot\vec{a} + T + \max_{X^b} \quad \{U(X^b) - X^b\} + \max_{\vec{a}_{+1}} \quad \{-\vec{1}\cdot\vec{a}_{+1} + \beta V_s(\vec{a}_{+1}) - \mathbb{1}_{\{d_{+1}>0\}} \times f - \mathbb{1}_{\{e_{+1}>0\}} \times \delta\}, \tag{1}
$$

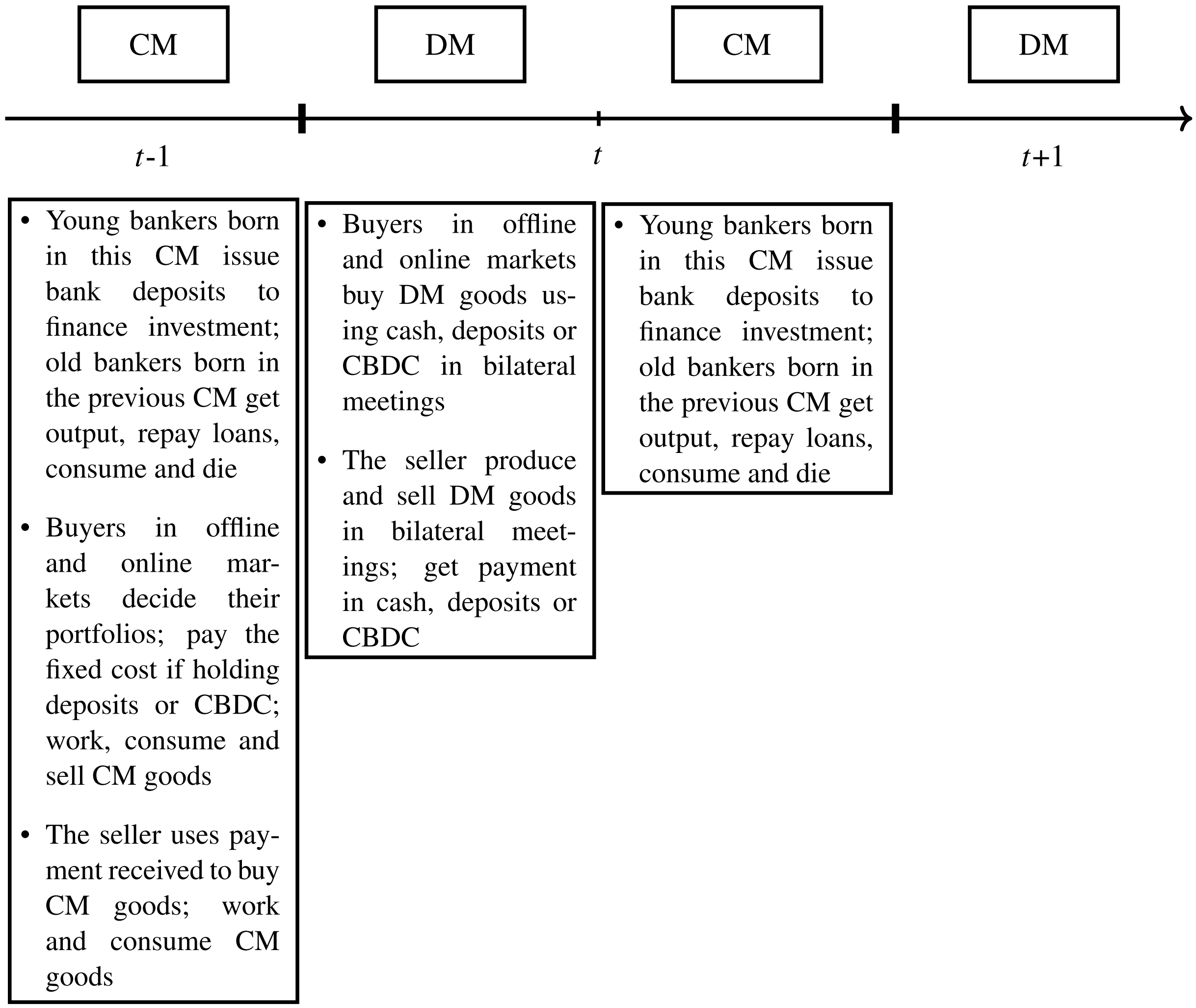

Figure 1: Timeline

subject to $e_{+1} \leq x$.

Buyers' DM value function is

$$\begin{aligned} V_s(\vec{a}) &= u(q_s) + W_s(\vec{a} - \vec{p}) \\ &= u(q_s) + R \cdot (\vec{a}_s - \vec{p}_s) + W_{s,+1}(0) \end{aligned}$$

where $(\vec{p}_s \equiv (p_s(c), p_s(d), p_s(e)), q_s)$ are the terms of trade, representing payment in cash, deposits and CBDC, and the amount of DM goods traded in market s.

Given that a buyer brings portfolio $\vec{a}_s = (c, d, e)$ into the DM in market s, the terms of trade assuming TIOLI offer trading protocol solves the following problem:

$$\begin{aligned} \max_{q_s,\vec{p}_s} \quad & S^b \\ \text{subject to:} \quad & S^s \geq 0 \\ & 0 \leq \vec{p}_s \leq \vec{a}_s, \end{aligned}$$

where $S^b = u(q_s) + W_s(\vec{a} - \vec{p}) - W_s(\vec{a}) = u(q_s) - R \cdot \vec{p}_s$, and $S^s = -q_s + R \cdot \vec{p}_s$. S^b and S^s are buyers' and sellers' surplus from DM trade respectively. Specifically, a buyer's trade surplus is simply her gain from DM consumption minus the value of payment transferred.

Solutions to the above problem is

$$q_s = R \cdot \vec{p}_s = \begin{cases} R \cdot \vec{a}_s & ,\text{for} \quad R \cdot \vec{a}_s < q^*, \\ q^* & ,\text{for} \quad R \cdot \vec{a}_s \geq q^*. \end{cases} \tag{2}$$

With the above solutions, buyers' CM value function can be reexpressed as follows

$$\begin{aligned} W_s(\vec{a}) =& R \cdot \vec{a} + T + \max_{X^b} \quad \{U(X^b) - X^b\} + \max_{\vec{a}_{s,+1}} \quad \{-\vec{1} \cdot \vec{a}_{s,+1} + \beta(u(q_{s,+1}) + R \cdot \vec{a}_{s,+1} - q_{s,+1}) \\ & - \mathbb{1}_{\{d_{+1}>0\}} \times f - \mathbb{1}_{\{e_{+1}>0\}} \times \delta\} + \beta W_{s,+1}(0), \end{aligned}$$

subject to $e_{+1} \leq x$. The Lagrangian for the buyers' portfolio problem above is

$$\begin{aligned} & \mathscr{L} = -\vec{1} \cdot \vec{a}_{s,+1} + \beta(u(q_{s,+1}) + R \cdot \vec{a}_{s,+1} - q_{s,+1}) - \mu(e_{+1} - x) \\ \text{subject to} \quad & e_{+1} \leq x. \end{aligned}$$

Define function $\lambda(L) \equiv \max\{u'(L) - 1, 0\}$, where L is available liquidity. The first-order conditions dictating households' demand for cash, deposits and CBDC are

$$\lambda(R \cdot \vec{a}_1) \leq \frac{1}{\beta R^c} - 1, \tag{3}$$

$$\lambda(R \cdot \vec{a}_2) \leq \frac{1}{\beta R^d} - 1, \tag{4}$$

$$\lambda(R \cdot \vec{a}_s) \leq \frac{1+\mu}{\beta R^e} - 1, \tag{5}$$

and the complementary slackness conditions are

$$\mu \geq 0, \tag{6}$$

$$\mu(e_{+1} - x) = 0. \tag{7}$$

2.4 Asset supply

2.4.1 Currency

I focus on stationary equilibria where the supply of central bank monies, including both paper money and CBDC, grows at a constant rate π such that $\frac{M}{M_{-1}} = \frac{\phi_{-1}}{\phi} = \pi$. The central bank controls only the total supply of cash and CBDC, but not its composition. Note that cash and CBDC have the same price. In other words, the exchange rate between cash and CBDC is fixed at one. Households can exchange one type of central bank money for another on a one-to-one basis, should they wish to. Therefore, the central bank's budget constraint is $\phi(C+E)+\phi R=\phi(C_{-1}+(1+i)E_{-1})+\phi R_{-1}+\tau+\mathbb{1}_{\{E>0\}}\times F-(\lambda_1\lambda_1^e+\lambda_2\lambda_2^e)\delta_2$, where C, E and R are the amounts of cash, CBDC and reserves outstanding, τ is the real transfer to households, F is the per period cost of issuing CBDC borne by the central bank, and $(\lambda_1\lambda_1^e+\lambda_2\lambda_2^e)\delta_2$ is the amount of cost recovered by collecting fees from households using CBDC where λ_1^e and λ_2^e are fractions of offline and online transactions where CBDC is used.

2.4.2 Bank deposits

The banking sector is charging a single deposit rate. Let $\hat{\gamma}$ denote the productivity of the cutoff bank whose pledgeable future income is just enough to cover the promised repayment on deposits. The pledgeability restriction is only placed on returns of project investment. Banks' asset holding of central bank reserves is fully pledgeable. And it is assumed that bankers' ability to issue deposits is restricted by this plegeability constraint. Thus I have

$$R^d=(1-\mu)\varepsilon\hat{\gamma}+\frac{\mu}{\pi} \tag{8}$$

The right-hand side is the cutoff bank's pledgeable income, which has two components. For every unit of deposit issued, $(1-\mu)$ units is invested in the project which generates $(1-\mu)\hat{\gamma}$ units of CM goods next period and only a fraction ε of this return can be pledged. This explains the first term. The remaining μ units of issued deposits is invested in reserves with a rate of return of $\frac{1}{\pi}$, which is the second term.

2.5 Market clearing

Since cash is only used as a means of payment in the offline market and deposits are only used in the online market, their market clearing conditions are

$$\phi C=\lambda_1\lambda_1^c c, \tag{9}$$

$$\eta(\bar{\gamma}-\hat{\gamma})=\lambda_2\lambda_2^d d, \tag{10}$$

where λ_1^c is the fraction of meetings in the offline market where cash is used, and λ_2^d is the fraction of online meetings where deposits are used. The market clearing condition for CBDC is

$$\phi E=(\lambda_1\lambda_1^e+\lambda_2\lambda_2^e)e, \tag{11}$$

where $\lambda_1\equiv\lambda_1^c+\lambda_1^e$, $\lambda_2\equiv\lambda_2^d+\lambda_2^e$, and $\lambda_1+\lambda_2=1$. The market clearing condition for reserves is simply

$$R=R_{-1} \tag{12}$$

Definition 1. *A stationary equilibrium is a list of portfolios accumulated by buyers* $\{\vec{a}_s\}_{s=1,2}$*, terms of trade* $\{\vec{p}_s, q_s\}_{s=1,2}$*, fractions of buyers holding each medium of exchange in the offline and online markets* $\{\lambda_1^c, \lambda_1^e, \lambda_2^d, \lambda_2^e\}$*, the deposit rate* R^d *and the cutoff* $\hat{\gamma}$ *that satisfy equations* (2)-(11) *given policy parameters* $\{\pi, R^c, R^e, \delta_2, x\}$.

In the following section, I will characterize the equilibrium in the benchmark economy where there is no CBDC issuance. Then, I will introduce CBDC into the economy and derive households' portfolio choices for both cases when the holding limit on CBDC is not binding and is binding, and for the case where another policy tool is introduced such that households are entitled with a lower fixed cost of using CBDC when they choose to hold a certain amount of CBDC in Section 4, and elaborate on the welfare properties of some interesting equilibria in Section 5.

3 Equilibrium without CBDC

When CBDC is not introduced, buyers use cash in the offline market and deposits in the online market. Therefore, the corresponding first-order conditions for cash and deposits will hold with equality:

$$\lambda(R^c c(R^c)) = \frac{1}{\beta R^c} - 1, \text{and} \tag{13}$$

$$\lambda(R_0^d d(R_0^d)) = \frac{1}{\beta R_0^d} - 1, \tag{14}$$

where the subscript 0 indicates the deposit rate in the benchmark equilibrium without CBDC.

Using the market clearing conditions, conditions (13)-(14) can be reexpressed as

$$\lambda(\frac{c}{\pi}) = \frac{\pi}{\beta} - 1 \tag{15}$$

$$\lambda(R_0^d \frac{\eta(\bar{\gamma} - \frac{R_0^d - \frac{\mu}{\pi}}{(1-\mu)\varepsilon})}{\lambda_2}) = \frac{1}{\beta R_0^d} - 1 \tag{16}$$

To guarantee the existence of equilibrium in the online market where deposits are used, I assume preferences are such that $d(R_0^d) = \frac{\lambda^{-1}(\frac{1}{\beta R_0^d} - 1)}{R_0^d}$ is strictly increasing in R_0^d. The following proposition characterizes the equilibrium with no CBDC.

Definition 2. *The unique equilibrium in the benchmark economy without CBDC consists of portfolios* $\{\vec{a}_1 = (c, 0, 0), \vec{a}_2 = (0, d(R_0^d), 0)\}$*, terms of trade* $\{\vec{p}_s, q_s\}_{s=1,2}$*, and a deposit rate* R_0^d *satisfying equations* (2) *and* (13)-(16) *given policy parameters* $\{\pi, R^c\}$.

4 Equilibrium with CBDC

Once CBDC is introduced, buyers can choose between two means of payment in both offline and online markets since it is assumed that CBDC can be used universally. Their choices depend critically on parameters controlled by the central bank such as rates of return on paper and digital public monies, the holding limit on CBDC, and the fee charged by the central bank for using CBDC: R^c, R^e, x, δ_2. The design option of the holding limit is motivated by the fact that central bankers, including those in Europe, the United Kingdom and China, are considering or have already implemented it trying to minimize the potential negative impact of introducing a state-backed digital money on private intermediaries. And since maintaining CBDC in the economy is costly, central bankers may seek to recover part of their expenses by charging fees to the public in exchange for providing a safe and efficient digital payment option. Alternatively, the central bank may promote CBDC usage by offering subsidies to users. These considerations give rise to the design option of the CBDC fee, δ_2. I will derive buyers' optimal portfolio choices given different designs of CBDC in the following subsections. First, the rate of return on CBDC is sufficently low or the holding limit on CBDC is sufficiently high such that the amount of CBDC that households carry, if they use it, does not exceed the limit. And the opposite for the second scenario. In the third scenario, when the holding limit on CBDC is not binding, the central bank creates another tool trying to see if they can incentivize households to hold more CBDC by subsidizing the use of a certain amount of CBDC while not undermining the deposit creation of private banks.

4.1 CBDC's holding limit x is not binding

Under this scenario, $x \geq e(R^e)$, where $e(R^e)$ is the amount of CBDC that buyers will accumulate in the CM when only CBDC is held and its rate of return is R^e. Specifically, $e(R^e)$ is given by the first-order condition: $\lambda(R^e e(R^e)) = \frac{1}{\beta R^e} - 1$.

In the offline market, buyers choose between cash and CBDC whichever has a higher CM value function. It will never be optimal for buyers to accumulate both media of exchange in a single portfolio under this scenario where the holding limit on CBDC is not binding since the only feasible case is when the two types of monies have the same rates of return, and when this is the case, buyers will optimally choose to bring only cash given the strictly positive fixed cost of using CBDC. The following CM value functions for bringing only cash and only CBDC are obtained by rearranging and collecting terms from expression (1).

$$W_1^c = (\beta R^c - 1)c(R^c) + \beta(u(q_1^c) - q_1^c) + A, \tag{17}$$
$$W_1^e = (\beta R^e - 1)e(R^e) + \beta(u(q_1^e) - q_1^e) - (\delta_1 + \delta_2) + A, \tag{18}$$

where $A = R \cdot \vec{a} + T + \max_{X^b}\{U(X^b) - X^b\} + \beta W_{1,+1}(0)$ is independent of buyers' choice between cash and CBDC. The superscripts c and e indicate the types of money held: cash and CBDC respectively. Similar to $e(R^e)$, $c(R^c)$ is pinned down by the first-order condition derived in the previous section: $\lambda(R^c c(R^c)) = \frac{1}{\beta R^c} - 1$.

With the two CM value functions (17)-(18), I obtain a threshold of R^e, which depends on policy parameters R^c and δ_2, and I denote it as $R^e(R^c, \delta_2)$. $R^e(R^c, \delta_2)$ is the rate of return on CBDC that makes it as desirable as cash to buyers. Given the strictly positive fixed cost of using CBDC, CBDC will have to offer a higher rate of return to incentivize buyers to switch from using cash to its digital counterpart. The following proposition dictates offline buyers' portfolio choices when the rate of return on CBDC takes different ranges of value. Proofs of all propositions and lemmas are contained in the Appendix.

Proposition 1. *When $R^e > R^e(R^c, \delta_2)$, buyers will choose to accumulate CBDC only ; when $R^e < R^e(R^c, \delta_2)$, buyers will choose to accumulate cash only. Buyers are indifferent between accumulating CBDC and cash when $R^e = R^e(R^c, \delta_2)$, in which case the fractions of households carrying only cash and only CBDC in the offline market, λ_1^c and λ_1^e, are indeterminate.*

Similarly, in the online market, buyers choose between deposits and CBDC whichever has a higher CM value function. It still will not be optimal for buyers to hold a mixed portfolio of the two since when they have the same rates of return, buyers will choose the one with a lower fixed cost. By rearranging and collecting terms from expression (1), I have

$$W_2^d(R_0^d) = (\beta R_0^d - 1)d(R_0^d) + \beta(u(q_2^d(R_0^d)) - q_2^d(R_0^d)) - (f_1 + f_2) + B, \tag{19}$$

$$W_2^e = (\beta R^e - 1)e(R^e) + \beta(u(q_2^e) - q_2^e) - (\delta_1 + \delta_2) + B, \tag{20}$$

where $B = R \cdot \vec{a} + T + \max_{X^b}\{U(X^b) - X^b\} + \beta W_{2,+1}(0)$. The superscript d imply that only deposits are held in the portfolio. A key difference from the offline market is that when CBDC offers a higher CM value function than deposits, CBDC will not replace deposits completely in the online market. Instead, the deposit rate R_0^d will increase to R_1^d until the CM value function of holding deposits matches the CM value function of holding CBDC given that some bankers are sufficiently productive to remain profitable even when borrowing from households at this higher interest rate. Therefore, with the two CM value functions (19)-(20), I obtain a threshold of R^e, which depends on R_0^d and δ_2, and I denote it as $R^e(R_0^d, \delta_2)$. $R^e(R_0^d, \delta_2)$ is the rate of return on CBDC that makes it as desirable as deposits in the benchmark economy to buyers.

Proposition 2. *When $R^e > R^e(R_0^d, \delta_2)$, the deposit rate will increase until deposits coexist with CBDC at the extensive margin in the online market where a fraction $\lambda_2^d(R_1^d)$ of buyers will hold deposits, while the remaining fraction, $\lambda_2^e(R_1^d) \equiv 1 - \lambda_2^d(R_1^d)$, of buyers will hold CBDC; when $R^e \leq R^e(R_0^d, \delta_2)$, buyers will choose to accumulate deposits and CBDC will not be adopted in the online market.*

4.2 CBDC's holding limit x is binding

Under this scenario, $x < e(R^e)$. The central bank now has all the three CBDC-related policy tools at its disposal: the interest rate, the holding limit, and the fee. More importantly, buyers may choose to hold both means of payment in a single portfolio. The intuition is that when the rate of return on CBDC is high and the fixed cost of holding CBDC is low, buyers will start accumulating CBDC up to the holding limit, and when the holding limit x is low, buyers will top up with cash or deposits to explore gains from more DM trade.

Therefore, in the offline market, there are three possible types of portfolios that buyers can choose from. Buyers may hold only cash, hold CBDC up to the holding limit, or hold CBDC up to the limit and supplement it with cash. Again, buyers will choose to hold the portfolio with the highest CM value function. The CM value functions for these three types of portfolios are:

$$
\begin{aligned}
W_1^c &= (\beta R^c - 1)c(R^c) + \beta(u(q_1^c) - q_1^c) + A, && (21)\\
W_1^x &= (\beta R^e - 1)x + \beta(u(q_1^x) - q_1^x) - (\delta_1 + \delta_2) + A, && (22)\\
W_1^{cx} &= (\beta R^e - 1)x + (\beta R^c - 1)\tilde{c} + \beta(u(q_1^c) - q_1^c) - (\delta_1 + \delta_2) + A \\
&= W_1^c + (\frac{R^e}{R^c} - 1)x - (\delta_1 + \delta_2), \tilde{c} \equiv \frac{R^c c(R^c) - R^e x}{R^c}, && (23)
\end{aligned}
$$

where the superscript x indicates that buyers hold x units of CBDC in the portfolio and cx indicates a mixed portfolio. The third portfolio $(\tilde{c}, 0, x)$, where cash coexists with CBDC at the intensive margin, is feasible only when $R^e > R^c$ and $R^e x < R^c c(R^c)$. The former condition implies that buyers will start accumulating CBDC first which has a higher rate of return if holding CBDC is optimal, while the latter one means that the marginal gain from more DM trade exceeds the marginal cost of topping up with cash. And when the third portfolio is feasible, $W_1^{cx} > W_1^x$ because the gains from increased DM trade outweigh the costs of holding more money. In other words, when cash can coexist with CBDC at the intensive margin, buyers will never choose the CBDC-only portfolio. The following lemma summarizes this result.

Lemma 1. *When $R^e > R^c$ and $R^e x < R^c c(R^c)$, buyers will never choose the CBDC-only portfolio.*

It is important to note that the amount of DM goods traded is the same for the first and third portfolios where cash is held. However, by accumulating CBDC first and then topping up with cash in the third mixed portfolio, buyers can benefit from potentially lower cost of accumulating money when CBDC has a higher interest rate than cash by paying the fixed cost of using CBDC.

Next, I will derive buyers' optimal portfolio choices in the offline market for the following three cases. To facilitate the analysis, I define three cutoffs of x: $x_1 \equiv \frac{\delta_1+\delta_2}{\frac{R^e}{R^c}-1}$, $x_2 \equiv x(R^c, R^e, \delta_2)$, and $x_3 \equiv \frac{R^c c(R^c)}{R^e}$, where x_2 equates the CM value functions of the cash-only and CBDC-only portfolios. First, when CBDC is not paying a higher interest rate than cash, CBDC will not be adopted given the fixed cost of using it. Second, when the rate of return on CBDC is higher than that on cash and the holding limit of CBDC is sufficiently large such that $x \geq x_3$, buyers will choose between the cash-only and CBDC-only portfolios which has a higher CM value function. Third, when the rate of return on CBDC exceeds that on cash and the holding limit of CBDC is small such that $x < x_3$, buyers will choose between the cash-only and the mixed portfolios. When savings on the cost of holding money by substituting x units of cash with CBDC surpass the fixed cost of holding CBDC such that $x > x_1$, buyers will opt for the third mixed portfolio. The following proposition encapsulates the results discussed above.

Proposition 3. *in the offline market, buyers prefer the cash-only portfolio when $R^e \leq R^c$, or $R^e > R^c$, $R^e x \geq R^c c(R^c)$ and $x < x_2$, or $R^e > R^c$, $R^e x < R^c c(R^c)$ and $x < x_1$; prefer the CBDC-only portfolio when $R^e > R^c$, $R^e x \geq R^c c(R^c)$ and $x > x_2$; prefer the mixed portfolio when $R^e > R^c$, $R^e x < R^c c(R^c)$ and $x > x_1$.*

With these results, I can trace out buyers' optimal portfolio choices in the offline market as x changes. It can be shown that when $x_1 < x_3$, then $x_2 < x_3$, and when $x_1 \geq x_3$, then $x_2 \geq x_3$. Now, suppose $R^e > R^c$ and $x_1 < x_3$. Buyers will hold only cash when $x < x_1$; hold both cash and CBDC when $x_1 < x < x_3$; hold only CBDC when $x > x_3$. Suppose $R^e > R^c$ and $x_1 \geq x_3$. The condition $x_1 \geq x_3$ implies that the range of holding limits making the mixed portfolio optimal lies outside the feasible set of the mixed portfolio. Therefore, buyers optimally choose between the first two portfolios. Buyers will choose the all-cash portfolio when $x < x_2$; choose the all-CBDC portfolio when $x > x_2$. Figures 2 and 3 summarizes buyers' optimal portfolio choices in the two cases.

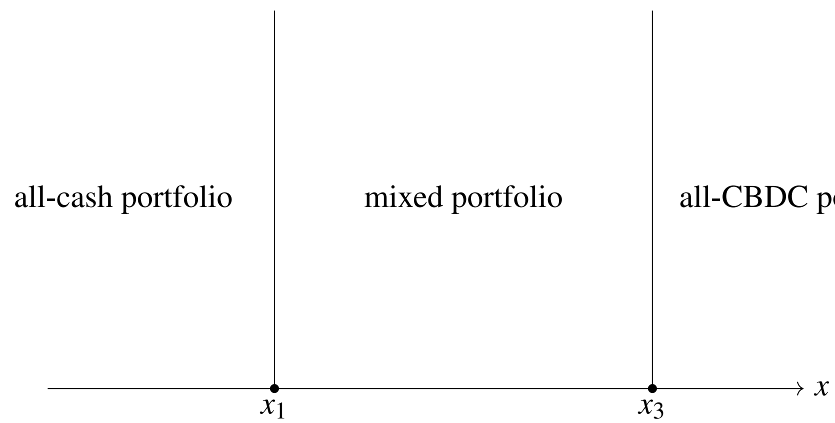

Figure 2: Optimal portfolio in the offline market when $R^e > R^c$ and $x_1 < x_3$.

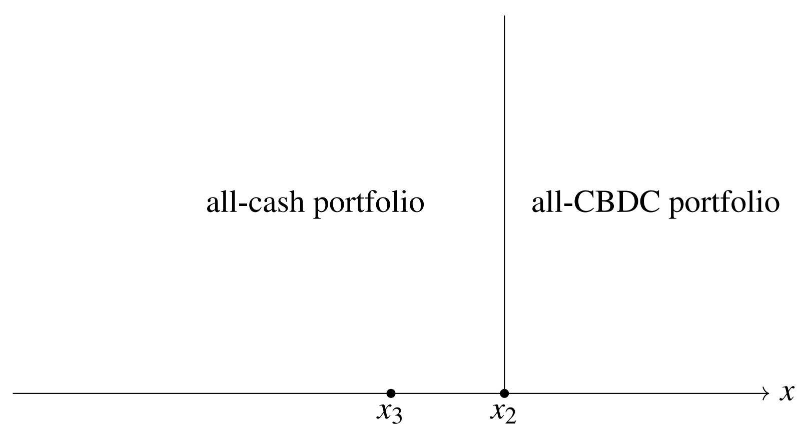

Figure 3: Optimal portfolio in the offline market when $R^e > R^c$ and $x_1 \geq x_3$.

Similarly, in the online market, there are also three possible types of portfolios that buyers can hold. Buyers may hold only deposits, hold only CBDC up to the holding limit, or hold CBDC up to the holding limit and then top up with deposits. The CM value functions of the three types of portfolios are:

$$W_2^d(R_0^d) = (\beta R_0^d - 1)d(R_0^d) + \beta(u(q_2^d(R_0^d)) - q_2^d(R_0^d)) + B, \tag{24}$$

$$W_2^x = (\beta R^e - 1)x + \beta(u(q_2^x) - q_2^x) - (\delta_1 + \delta_2) + B, \tag{25}$$

$$W_2^{dx}(R^{d*}) = (\beta R^e - 1)x + (\beta R^{d*} - 1)\tilde{d} + \beta(u(q_2^{d*}) - q_2^{d*}) - (\delta_1 + \delta_2) + B$$
$$, \tilde{d} \equiv \frac{R^{d*}d(R^{d*}) - R^e x}{R^{d*}}, \tag{26}$$

where R^{d*} is the equilibrium deposit rate when online buyers hold the mixed portfolio and it is determined by the deposit market clearing condition: $\eta(\bar{\gamma} - \hat{\gamma}(R^{d*})) = \lambda_2(d(R^{d*}) - \frac{R^e x}{R^{d*}})$. This condition defines R^{d*} as a function of $R^e x$, and I denote it as $R^{d*}(R^e x)$. When $R^e x$ is strictly positive, $R^{d*} > R_0^d$.

The mixed portfolio $(0, \tilde{d}, x)$, where deposits coexist with CBDC at the intensive margin, is feasible only when

$$R^e > R^{d*} \quad \text{and} \quad R^e x < R^{d*}d(R^{d*}). \tag{27}$$

Since feasible portfolios are different when the two conditions in expression (27) are not met and are met, I will derive buyers' portfolio choices under the two cases respectively. When conditions (27) are not met, buyers choose between the all-deposits and all-CBDC portfolios. If $W_2^x \leq W_2^d(R_0^d)$, CBDC will not be adopted in the online market. If $W_2^x > W_2^d(R_0^d)$, the deposit rate R_0^d will increase to R_x^d until $W_2^d(R_x^d) = W_2^x$, and deposits coexist with CBDC at the extensive margin. When conditions (27) are met, buyers have more choices and can choose among all the three portfolios. If $W_2^d(R_0^d) > \max\{W_2^x, W_2^{dx}(R^{d*})\}$, CBDC will not be used in the online market; if $W_2^{dx}(R^{d*}) > \max\{W_2^x, W_2^d(R_0^d)\}$, CBDC coexists with deposits at the intensive margin; if $W_2^x > \max\{W_2^{dx}(R^{d*}), W_2^d(R_0^d)\}$, again the deposit rate R_0^d will increase to R_x^d and deposits coexist with CBDC at the extensive margin. **better presentation of these cases**

Unlike the equilibria in the offline market, the transition path across equilibria as x changes in the online market is less obvious since it depends on R^{d*}, which in equilibrium depends on $R^e x$ and the expression of the function $R^{d*}(R^e x)$ is implicit. In what follows, I show how some important equilibrium objects respond to changes in the three CBDC-related parameters: R^e, x, and δ_2.

When deposits and CBDC coexist at the extensive margin in equilibrium, the fraction of online households holding deposits decreases as R^e increases, x increases, and δ_2 decreases. This is because a higher deposit rate, which makes deposits as appealing as a more attractive CBDC to buyers, reduces the supply of deposits from private bankers while increasing the demand for deposits among households who use them. When deposits and CBDC coexist at the intensive margin in equilibrium, the equilibrium deposit rate R^{d*} is strictly increasing in $R^e x$ so that the quantity of DM goods traded moves in tandem with the attractiveness of CBDC, and independent of the CBDC fee, δ_2, since the fee is considered as a sunk cost after the decision to accumulate CBDC has been made.

4.3 Differentiated CBDC fee: κ

Now, to encourage households to accumulate more CBDC while minimizing the disintermediation risk, which cannot be achieved by the three policy tools examined so far, the central bank introduces another tool: κ. When the holding limit x is not binding, if households hold at least κ units of CBDC, the central bank will charge them a lower bank fee or even provide them with a subsidy for using CBDC such that the central bank-controlled component of the fixed cost is $\tilde{\delta}_2$, which is smaller than δ_2. I will first derive the amount of CBDC that households choose to accumulate if they were to hold CBDC given this policy stimulus. Then I will determine the equilibria when κ takes different values.

When κ is not binding such that $\kappa \leq e_{+1}(R^e)$, households enjoy the lower CBDC fee or subsidy $\tilde{\delta}_2$ by simply holding the amount of CBDC satisfying the corresponding first-order condition and therefore there is no distortion regarding the amount of CBDC held by buyers. However, when κ is binding such that $\kappa > e_{+1}(R^e)$, households have to decide whether they want to accumulate κ units of CBDC and enjoy $\tilde{\delta}_2$ or rather stick to $e_{+1}(R^e)$ and bear δ_2. Denote households' choice of the amount of CBDC to accumulate as e_{+1} and the CM value functions in offline and online markets are summarized as

$$\begin{aligned} W_s(e_{+1}) =& (\beta R^e - 1)e_{s,+1} + \beta(u(q_s^e(e_{+1})) - q_s^e(e_{+1})) - \delta_1 \\ & - \mathbb{1}_{\{e_{s,+1} \geq \kappa\}} \times \tilde{\delta}_2 - \mathbb{1}_{\{e_{s,+1} < \kappa\}} \times \delta_2 + \mathbb{1}_{\{s=1\}} \times A + \mathbb{1}_{\{s=2\}} \times B \end{aligned} \tag{28}$$

Let

$$\begin{aligned} f(\kappa, R^e, \tilde{\delta}_2, \delta_2) \equiv & W_s(\kappa) - W_s(e_{+1}(R^e)) \\ & = (\beta R^e - 1)\kappa + \beta(u(q^\kappa) - q^\kappa) - (\beta R^e - 1)e_{+1}(R^e) \\ & - \beta(u(q(e_{+1}(R^e))) - q(e_{+1}(R^e))) - \tilde{\delta}_2 + \delta_2, \end{aligned} \tag{29}$$

where the superscript κ indicate that households hold κ units of CBDC in the portfolio. It can be shown that $f(\kappa, R^e, \tilde{\delta}_2, \delta_2)$ is strictly decreasing in κ, therefore, I can obtain the threshold $\kappa(R^e)$ as a function of R^e from it. When κ is binding and households decide to hold this higher amount of CBDC than the amount that gives them the highest utility when they make their portfolio decisions in the CM, they experience a loss by deviating from the optimum. Meanwhile, they also benefit from a lower fixed cost of using CBDC. The larger κ is, the further they deviate from the optimum, and the less likely they are to overaccumulate CBDC. When κ falls below the threshold $\kappa(R^e)$, the benefit of overaccumulating CBDC outweighs the cost, and therefore households will choose to hold κ units of CBDC. On the contrary, when κ exceeds the threshold, the benefit no longer compensates for the cost. Thus, I have the following lemma.

Lemma 2. *The amount of CBDC that households choose to accumulate if they were to hold CBDC, e_{+1}, takes the following function:*

$$e_{+1} = \begin{cases} e_{+1}(R^e) & ,for \quad \kappa \leq e_{+1}(R^e) \\ \kappa & ,for \quad e_{+1}(R^e) < \kappa \leq \kappa(R^e) \\ e_{+1}(R^e) & ,for \quad \kappa > e_{+1}(R^e), \kappa > \kappa(R^e) \end{cases} \tag{30}$$

Now, I can characterize the types of equilibria in both offline and online markets. in the offline market, buyers choose between cash and CBDC whichever has a higher CM value function.

$$W_1^c = (\beta R^c - 1)c(R^c) + \beta(u(q_1^c) - q_1^c) + A, \tag{31}$$

$$W_1^e = \begin{cases} (\beta R^e - 1)e(R^e) + \beta(u(q_1^e) - q_1^e) - (\delta_1 + \tilde{\delta}_2) + A & \text{,for} \quad \kappa \le e_{+1}(R^e) \\ (\beta R^e - 1)\kappa + \beta(u(q_1^\kappa) - q_1^\kappa) - (\delta_1 + \tilde{\delta}_2) + A & \text{,for} \quad e_{+1}(R^e) < \kappa \le \kappa(R^e) \\ (\beta R^e - 1)e(R^e) + \beta(u(q_1^e) - q_1^e) - (\delta_1 + \delta_2) + A & \text{,for} \quad \kappa > \kappa(R^e) \end{cases} \tag{32}$$

Since κ is the element of interest, I will only elaborate on the case where $e_{+1}(R^e) < \kappa \le \kappa(R^e)$. With the two CM value functions (31)-(32), I obtain a threshold of κ, which depends on R^c, R^e and $\tilde{\delta}_2$, and I denote it as $\kappa(R^c, R^e, \tilde{\delta}_2)$. $\kappa(R^c, R^e, \tilde{\delta}_2)$ is the amount of CBDC allowing households to enjoy a lower fixed cost that makes holding κ units of CBDC as desirable as holding cash to buyers.

Proposition 4. *Given $e_{+1}(R^e) < \kappa \le \kappa(R^e)$, when $\kappa < \kappa(R^c, R^e, \tilde{\delta}_2)$, buyers will choose to accumulate κ units of CBDC only; when $\kappa > \kappa(R^c, R^e, \tilde{\delta}_2)$, buyers will choose to accumulate cash only. Buyers are indifferent between accumulating κ units of CBDC and cash when $\kappa = \kappa(R^c, R^e, \tilde{\delta}_2)$.*

Similarly, in the online market, buyers compare the CM value functions of holding deposits and CBDC.

$$W_2^d(R_0^d) = (\beta R_0^d - 1)d(R_0^d) + \beta(u(q_2^d(R_0^d)) - q_2^d(R_0^d)) - (f_1 + f_2) + B \tag{33}$$

$$W_2^e = \begin{cases} (\beta R^e - 1)e(R^e) + \beta(u(q_2^e) - q_2^e) - (\delta_1 + \tilde{\delta}_2) + A & \text{,for} \quad \kappa \le e_{+1}(R^e) \\ (\beta R^e - 1)\kappa + \beta(u(q_2^\kappa) - q_2^\kappa) - (\delta_1 + \tilde{\delta}_2) + A & \text{,for} \quad e_{+1}(R^e) < \kappa \le \kappa(R^e) \\ (\beta R^e - 1)e(R^e) + \beta(u(q_2^e) - q_2^e) - (\delta_1 + \delta_2) + A & \text{,for} \quad \kappa > \kappa(R^e) \end{cases} \tag{34}$$

For the intermediate case, I have the threshold $\kappa(R_0^d, R^e, \tilde{\delta}_2)$, which is the value that κ should take to make holding κ units of CBDC as desirable as holding deposits in the benchmark economy to buyers.

Proposition 5. *Given $e_{+1}(R^e) < \kappa \le \kappa(R^e)$, when $\kappa \ge \kappa(R_0^d, R^e, \tilde{\delta}_2)$, buyers will choose to accumulate deposits only and CBDC will not be adopted in the online market; when $\kappa < \kappa(R_0^d, R^e, \tilde{\delta}_2)$, the deposit rate will increase to R_κ^d, and CBDC coexists with deposits at the extensive margin, in which case a fraction $\lambda_2^d(R_\kappa^d)$ of buyers will hold deposits, while the remaining fraction, $\lambda_2^e(R_\kappa^d) \equiv 1 - \lambda_2^d(R_\kappa^d)$, of buyers will hold κ units of CBDC.*

The following proposition shows when CBDC and deposits coexist at the extensive margin in equilibrium, the central bank can increase households' use of CBDC at the intensive margin while crowding in deposits.

Proposition 6. *As κ increases, the CM value function of holding κ units of CBDC decreases, leading to a decrease in R_κ^d and less disintermediation.*

5 Welfare analysis

In this section, I will take a local perspective and characterize the optimal design of CBDC in terms of the interest rate, the holding limit, the fixed cost, and the differentiated CBDC fee scheme within a specific market as well as within equilibrium. It is difficult to analyze the optimal choices maximizing the social welfare function across equilibria globally since the impact of CBDC design options on households' portfolio decisions in the online market is less straightforward compared to the offline market. Thus, in what follows, I first examine the partial welfare function in the offline market when the holding limit on CBDC is binding, and then discuss the welfare impact of the four CBDC design parameters in equilibria where CBDC coexists with deposits at the intensive and extensive margin. In particular, I want to see the mechanism through which these policy tools affect welfare, whether there is redundancy among them in the sense that a subset of them can be viewed as single tool, and will the answer change across different equilibria.

5.1 Partial welfare function in the offline market

First, if we only focus on the offline market, ignoring the online market and bankers' investment, the partial welfare functions of the three types of equilibria when x is binding are:

$$\mathscr{W}_1^c = u(q_1^c) - q_1^c, \tag{35}$$

$$\mathscr{W}_1^x = u(q_1^x) - q_1^x - \frac{\delta_1}{\beta}, \tag{36}$$

$$\mathscr{W}_1^{cx} = u(q_1^c) - q_1^c - \frac{\delta_1}{\beta}. \tag{37}$$

As derived in the previous section, when $R^e > R^c$ and $x_1 < x_3$, as the holding limit on CBDC increases, offline buyers will first hold cash only, then hold the mixed portfolio of cash and CBDC, and hold CBDC only when x is sufficiently large in equilibrium. The following lemma constrains the optimal holding limit that maximizes the partial welfare function.

Lemma 3. *Given $\mathscr{W}_1^{cx} < \mathscr{W}_1^c$ when $\delta_1 > 0$, and buyers hold the mixed portfolio when $x_1 < x < x_3$ conditioning on $R^e > R^c$ and $x_1 < x_3$, the optimal value of x maximizing the partial welfare function in the offline market will not fall within this intermediate range.*

Specifically, figure 4 plots the value of the partial welfare function in equilibrium against the holding limit on CBDC, x, when $\delta_1 > 0$, $R^e > R^c$ and $x_1 < x_3$. When $x = x_1$, in principle, offline buyers are indifferent between holding the cash-only portfolio and the mixed portfolio, although the latter results in lower partial welfare due to the fixed user cost of using CBDC. When $x > x_3$, offline buyers will hold only CBDC up to the limit, and the partial welfare function increases as the holding limit relaxes until it stops binding.

Proposition 7. *Conditioning on $\delta_1 > 0$, $R^e > R^c$ and $x_1 < x_3$, when $x > x_3$, buyers will hold the all-CBDC portfolio, and a sufficient condition for welfare improvement such that $\mathscr{W}_1^x > \mathscr{W}_1^c$ is $x \geq \frac{(\beta R^c - 1)c + \delta_2}{\beta R^e - 1}$.*

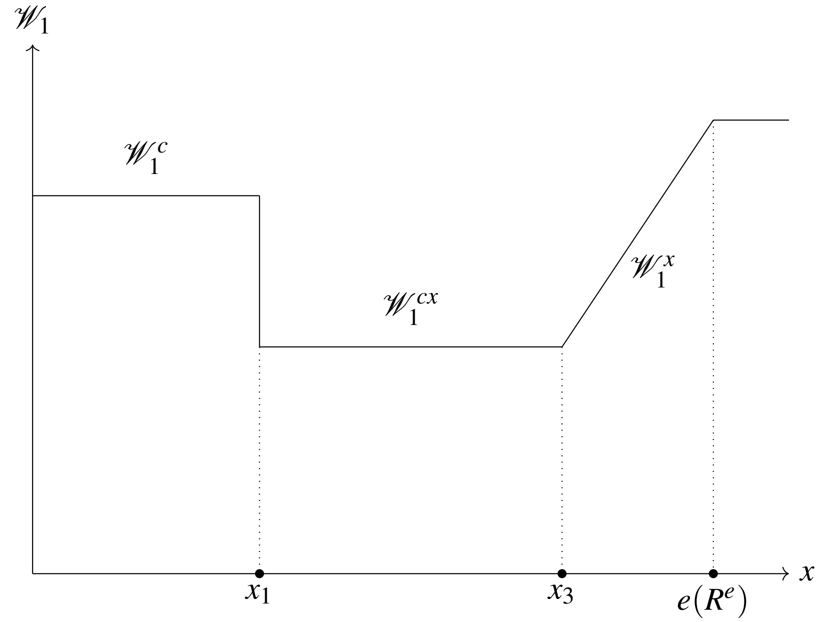

Figure 4: Partial welfare function in the offline market.

5.2 Benchmark economy without CBDC

In the benchmark economy where there is no CBDC, the aggregate welfare function is defined as $\mathscr{W} = \lambda_1(u(q_1^c) - q_1^c) + \lambda_2(u(q_2^d(R_0^d)) - q_2^d(R_0^d) - \frac{f_1}{\beta}) + \eta \int_{\hat{\gamma}(R_0^d)}^{\bar{\gamma}} (\gamma - \frac{1}{\beta}) d\gamma$, where the first two terms represent trade surplus in the offline and online market respectively, while the last one is the aggregate net output of bank investment. The last component highlights why introducing CBDC into the payment landscape can improve welfare. Even if the central bank can run the Friedman rule in both the offline and online markets such that $R^c = R_0^d = \frac{1}{\beta}$, there will still be underinvestment in the banking sector since the productivity of the marginal deposit-issuing bank is higher than the rate of time preference (i.e., $\hat{\gamma}(R_0^d = \frac{1}{\beta}) > \frac{1}{\beta}$). Furthermore, in cases of overinvestment in the benchmark economy, introducing CBDC to compete with private bank deposits enhances welfare. Therefore, in subsequent analyses, CBDC is introduced. In the following two equilibria, the CBDC holding limit x is binding and CBDC is used in both offline and online markets, making x an effective new policy tool for the central bank. I call the equilibrium where deposits coexist with CBDC at the intensive margin the intensive margin coexistence equilibrium, and the equilibrium where deposits coexist with CBDC at the extensive margin the extensive margin coexistence equilibrium.

5.3 Intensive margin coexistence equilibrium

In this equilibrium, offline buyers hold CBDC while online buyers hold both deposits and CBDC in their portfolios. The equilibrium deposit rate is R^{d*}, and the welfare function is

$$\mathscr{W} = \lambda_1(u(q_1^x) - q_1^x - \frac{\delta_1}{\beta}) + \lambda_2(u(q_2^{d*}) - q_2^{d*} - \frac{f_1+\delta_1}{\beta}) + \eta \int_{\hat{\gamma}(R^{d*})}^{\bar{\gamma}} (\gamma - \frac{1}{\beta}) d\gamma \tag{38}$$

It is obvious from the expression that the welfare function is a function of $R^e x$ and R^{d*}, where the latter is also a function of $R^e x$.

Lemma 4. *In this equilibrium, R^e and x are considered as a single policy tool in terms of their impact on welfare. Moreover, the welfare function is independent of the CBDC fee, δ_2.*

The welfare effect of a higher $R^e x$ is twofold. First, when q_1^x and q_2^{d*} are below the efficient DM trade q^*, increasing $R^e x$ has a positive trade effect by increasing the amount of DM trade in both offline and online markets, implied by the first two terms of the following partial derivative. Second, there is a disintermediation effect since a higher $R^e x$ results in a higher deposit rate. The last term reflects how investment or capital responds to changes in $R^e x$. If there was excessive investment before the policy change (i.e., $\hat{\gamma} < \frac{1}{\beta}$), disintermediation improves welfare. However, if there was underinvestment beforehand, crowding out deposits deteriorates welfare. Taking these effects together, raising $R^e x$ leads to higher welfare as long as the there is over-investment or the trade effect dominates the negative disintermediation effect.

$$\frac{\partial \mathscr{W}}{\partial (R^e x)} = \underbrace{\lambda_1(u'(q_1^x) - 1) + \lambda_2(u'(q_2^{d*}) - 1)\frac{\partial q_2^{d*}}{\partial R^{d*}}\frac{\partial R^{d*}}{\partial (R^e x)}}_{\text{trade effect}} + \underbrace{\frac{\eta(\frac{1}{\beta} - \hat{\gamma})}{(1-\mu)\varepsilon}\frac{\partial R^{d*}}{\partial (R^e x)}}_{\text{disintermediation effect}} \tag{39}$$

Proposition 8. *In the intensive margin coexistence equilibrium, raising $R^e x$ have a positive trade effect when q_1^x and q_2^{d*} are below q^*, and a positive or negative disintermediation effect.*

5.4 Extensive margin coexistence equilibrium

In this equilibrium, offline buyers hold CBDC; in the online market, some buyers hold deposits while others hold CBDC. The equilibrium deposit rate is R_x^d, and the welfare function is

$$\begin{aligned}\mathscr{W} =& \lambda_1(u(q_1^x) - q_1^x - \frac{\delta_1}{\beta}) + \lambda_2\{\lambda_2^d(u(q_2^d(R_x^d)) - q_2^d(R_x^d) - \frac{f_1}{\beta}) + \lambda_2^x(u(q_2^x) - q_2^x - \frac{\delta_1}{\beta})\} \\ &+ \eta \int_{\hat{\gamma}(R_x^d)}^{\bar{\gamma}} (\gamma - \frac{1}{\beta}) d\gamma \end{aligned} \tag{40}$$

Unlike the intensive margin coexistence equilibrium, here, R^e and x have different welfare impact. Again, the welfare function is a function of $R^e x$ and R_x^d, where the latter is a function of $R^e x$, x and δ_2.

Lemma 5. *In the extensive margin coexistence equilibrium, R^e and x have differentiated welfare impact arising from their differentiated impact on the equilibrium deposit rate R_x^d.*

The welfare impact of R^e, x, and δ_2 can be analyzed through the following partial derivatives. For policy tools R^e and x, aside from the trade effect and the disintermediation effect as in the intensive margin coexistence equilibrium, there is also a substitution effect. When R^e and x vary, the fractions of online households holding deposits and CBDC will also vary, and while the CM value functions are the same for the deposit-only and CBDC-only portfolios, the welfare is different. Therefore, the substitution effect depends on which portfolio provides a higher welfare. For policy tool δ_2, there are only trade effect for the deposit-only portfolio and the disintermediation effect.

$$\begin{aligned}\frac{\partial \mathscr{W}}{\partial R^e} = & \underbrace{(\lambda_1+\lambda_2\lambda_2^x)(u'(q_s^x)-1)x+\lambda_2\lambda_2^d(u'(q_2^d(R_x^d))-1)\frac{\partial q_2^d(R_x^d)}{\partial R_x^d}\frac{\partial R_x^d}{\partial R^e}}_{\text{trade effect}} \\ & \underbrace{+\lambda_2\left((u(q_2^d(R_x^d))-q_2^d(R_x^d)-\frac{f_1}{\beta})-(u(q_2^x)-q_2^x-\frac{\delta_1}{\beta})\right)\frac{\partial \lambda_2^d}{\partial R_x^d}\frac{\partial R_x^d}{\partial R^e}}_{\text{substitution effect}} \\ & + \underbrace{\frac{\eta(\frac{1}{\beta}-\hat{\gamma})}{(1-\mu)\varepsilon}\frac{\partial R_x^d}{\partial R^e}}_{\text{disintermediation effect}}\end{aligned} \tag{41}$$

$$\begin{aligned}\frac{\partial \mathscr{W}}{\partial x} = & \underbrace{(\lambda_1+\lambda_2\lambda_2^x)(u'(q_s^x)-1)R^e+\lambda_2\lambda_2^d(u'(q_2^d(R_x^d))-1)\frac{\partial q_2^d(R_x^d)}{\partial R_x^d}\frac{\partial R_x^d}{\partial x}}_{\text{trade effect}} \\ & \underbrace{+\lambda_2\left((u(q_2^d(R_x^d))-q_2^d(R_x^d)-\frac{f_1}{\beta})-(u(q_2^x)-q_2^x-\frac{\delta_1}{\beta})\right)\frac{\partial \lambda_2^d}{\partial R_x^d}\frac{\partial R_x^d}{\partial x}}_{\text{substitution effect}} \\ & + \underbrace{\frac{\eta(\frac{1}{\beta}-\hat{\gamma})}{(1-\mu)\varepsilon}\frac{\partial R_x^d}{\partial x}}_{\text{disintermediation effect}}\end{aligned} \tag{42}$$

$$\frac{\partial \mathscr{W}}{\partial \delta_2} = \underbrace{\lambda_2\lambda_2^d(u'(q_2^d(R_x^d))-1)\frac{\partial q_2^d(R_x^d)}{\partial R_x^d}\frac{\partial R_x^d}{\partial \delta_2}}_{\text{trade effect}} + \underbrace{\frac{\eta(\frac{1}{\beta}-\hat{\gamma})}{(1-\mu)\varepsilon}\frac{\partial R_x^d}{\partial \delta_2}}_{\text{disintermediation effect}} \tag{43}$$

5.5 κ equilibrium

In this equilibrium, κ is binding and CBDC is used in both offline and online markets. Specifically, in the online market, deposits coexist with CBDC at the extensive margin. This is true when $\kappa < min\{\kappa(R^e), \kappa(R^c, R^e, \tilde{\delta}_2), \kappa(R_0^d, R^e, \tilde{\delta}_2)\}$. The welfare function and the partial derivative with respect to κ are

$$\begin{aligned}\mathscr{W} =&\lambda_1(u(q_1^\kappa)-q_1^\kappa-\frac{\delta_1}{\beta})+\lambda_2\{\lambda_2^d(u(q_2^d(R_\kappa^d))-q_2^d(R_\kappa^d)-\frac{f_1}{\beta})+\lambda_2^\kappa(u(q_2^\kappa)-q_2^\kappa-\frac{\delta_1}{\beta})\}\\ &+\eta\int_{\hat{\gamma}(R_\kappa^d)}^{\bar{\gamma}}(\gamma-\frac{1}{\beta})d\gamma; \end{aligned} \tag{44}$$

$$\begin{aligned}\frac{\partial\mathscr{W}}{\partial\kappa} =&\underbrace{(\lambda_1+\lambda_2\lambda_2^\kappa)(u'(q_s^\kappa)-1)R^e}_{\text{trade effect in CBDC meetings}}+\underbrace{\lambda_2\lambda_2^d(u'(q_2^d(R_\kappa^d))-1)\frac{\partial q_2^d(R_\kappa^d)}{\partial R_\kappa^d}\frac{\partial R_\kappa^d}{\partial\kappa}}_{\text{trade effect in deposit meetings}}\\ &+\underbrace{\lambda_2\{(u(q_2^d(R_\kappa^d))-q_2^d(R_\kappa^d)-\frac{f_1}{\beta})-(u(q_2^\kappa)-q_2^\kappa-\frac{\delta_1}{\beta})\}\frac{\partial\lambda_2^d}{\partial R_\kappa^d}\frac{\partial R_\kappa^d}{\kappa}}_{\text{substitution effect}}\\ &+\underbrace{\frac{\eta(\frac{1}{\beta}-\hat{\gamma})}{(1-\mu)\varepsilon}\frac{\partial R_\kappa^d}{\partial\kappa}}_{\text{disintermediation effect}}.\end{aligned} \tag{45}$$

Unlike in previous equilibria, the trade effects in CBDC and deposit meetings move in opposite directions when κ changes.

6 Conclusion

This paper explores how the design features of a retail CBDC influence its coexistence with cash and bank deposits, its impact on financial intermediation, and its welfare implications. The findings suggest that a holding limit on CBDC can effectively mitigate disintermediation risks, enabling CBDC to coexist with other forms of money either at the intensive or extensive margin. Additionally, interest payments and holding limits can serve as substitutable tools for welfare optimization under certain equilibria, while a CBDC fee is only effective under specific conditions. To further encourage CBDC adoption without jeopardizing bank deposits, central banks can implement a minimum holding requirement tied to fee reductions or subsidies, fostering deposit retention while achieving policy objectives. These insights offer a theoretical foundation for central bank discussions on CBDC design, demonstrating the importance of nuanced policy measures to balance innovation with financial stability.

A Proofs

1. By setting $W_1^c = W_1^e \Leftrightarrow (\beta R^c-1)c(R^c)+\beta(u(q_1^c)-q_1^c)+A = (\beta R^e-1)e(R^e)+\beta(u(q_1^e)-q_1^e)-(\delta_1+\delta_2)+A \Leftrightarrow (\beta R^c-1)c(R^c)+\beta(u(R^c c(R^c))-R^c c(R^c)) = (\beta R^e-1)e(R^e)+\beta(u(R^e e(R^e))-R^e e(R^e))-(\delta_1+\delta_2) \Longrightarrow R^e(R^c,\delta_2)$.

2. When $R^e > R^e(R_0^d, \delta_2)$, the deposit rate R_0^d will increase to R_1^d until $W_2^d(R_1^d) = W_2^e$. The fraction of buyers holding deposits, $\lambda_2^d(R_1^d)$, is pinned down by the deposit market clearing condition: $\lambda_2^d = \frac{\eta(\bar{\gamma}-\hat{\gamma}(R_1^d))}{\lambda_2 d(R_1^d)}$. By assuming $\hat{\gamma}(R_1^d) < \bar{\gamma} < \frac{\lambda_2 d(R_1^d)}{\eta} + \hat{\gamma}(R_1^d)$, I have $0 < \lambda_2^d < 1$.

3. When $R^e > R^c$ and $R^e x \geq R^c c(R^c)$, buyers only choose between the first two portfolios. By setting $W_1^c = W_1^x$, I can obtain a threshold of x which depends on R^c, R^e and δ_2 and I denote it as $x(R^c, R^e, \delta_2)$. Buyers will hold cash when $x < x(R^c, R^e, \delta_2)$, hold CBDC when $x > x(R^c, R^e, \delta_2)$, and are indifferent between the two portfolios when $x = x(R^c, R^e, \delta_2)$. When $R^e > R^c$ and $R^e x < R^c c(R^c)$, buyers will only choose between the first and the third portfolios since the second one will never be optimal. By setting $W_1^c = W_1^{cx}$, I obtain another threshold for x with the explicit expression $\frac{\delta_1+\delta_2}{\frac{R^e}{R^c}-1}$. When $x > \frac{\delta_1+\delta_2}{\frac{R^e}{R^c}-1}$, savings on the cost of holding money by substituting x units of cash with CBDC surpass the fixed cost of holding CBDC, therefore, buyers will opt for the third mixed portfolio. Consequently, when $x < \frac{\delta_1+\delta_2}{\frac{R^e}{R^c}-1}$, buyers will opt for the cash-only portfolio, and buyers are just indifferent when $x = \frac{\delta_1+\delta_2}{\frac{R^e}{R^c}-1}$.

4. $x_1 \equiv \frac{\delta_1+\delta_2}{\frac{R^e}{R^c}-1}$, $x_2 \equiv x(R^c, R^e, \delta_2)$, $x_3 \equiv \frac{R^c c(R^c)}{R^e}$. When $x_1 < x_3 \Leftrightarrow \delta < \frac{R^e-R^c}{R^e}c$; since $W_1^x|_{x=x_2} = W_1^c$ and W_1^x is strictly increasing in x when x is binding, $x_2 < x_3 \Leftrightarrow W_1^x|_{x=x_3} > W_1^c \Leftrightarrow \delta < \frac{R^e-R^c}{R^e}c$. Thus, $x_1 < x_3 \Leftrightarrow x_2 < x_3$.

5. $\eta(\bar{\gamma} - \hat{\gamma}(R^{d*})) = \lambda_2(d(R^{d*}) - \frac{R^e x}{R^{d*}})$. Differentiate both sides with respect to $R^e x$, I have $-\frac{\eta}{(1-\mu)\varepsilon}\frac{\partial R^{d*}}{\partial R^e x} = \lambda_2\{\frac{\partial d(R^{d*})}{\partial R^{d*}}\frac{\partial R^{d*}}{\partial R^e x} - \frac{R^{d*}-R^e x\frac{\partial R^{d*}}{\partial R^e x}}{(R^{d*})^2}\} \Leftrightarrow \frac{\partial R^{d*}}{\partial R^e x} = \frac{1}{R^{d*}\{\frac{\partial d(R^{d*})}{\partial R^{d*}} + \frac{R^e x}{(R^{d*})^2} + \frac{\eta}{\lambda_2(1-\mu)\varepsilon}\}} > 0$.

6. $\frac{\partial f(\kappa, R^e, \tilde{\delta}_2, \delta_2)}{\partial \kappa} = \beta R^e - 1 + \beta R^e(u'(q^\kappa) - 1) < \beta R^e - 1 + \beta R^e(\frac{1}{\beta R^e} - 1) = 0$. By setting $f(\kappa, R^e, \tilde{\delta}_2, \delta_2) = 0$, I can obtain the threshold $\kappa(R^e)$ as a function of R^e. When $\kappa > \kappa(R^e) \Leftrightarrow W_s(\kappa) < W_s(e_{+1}(R^e)) \Leftrightarrow$ buyers will not be allured by $\tilde{\delta}_2$ and hold κ units of CBDC, but will hold whatever dictated by the first-order condition. When $\kappa \leq \kappa(R^e) \Leftrightarrow W_s(\kappa) \geq W_s(e_{+1}(R^e)) \Leftrightarrow$ buyers will choose to hold κ units of CBDC.

7. Proposition 7: When $x > x_3$, $W_1^x > W_1^c \Leftrightarrow \mathscr{W}_1^x + (\beta R^e - 1)x - (\beta R^c - 1)c(R^c) - \delta_2 > \mathscr{W}_1^c$.

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